

TRACES AND EMBEDDINGS OF ANISOTROPIC FUNCTION SPACES

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ABSTRACT. In this paper we characterize trace spaces of vector-valued Triebel-Lizorkin, Besov, Bessel-potential and Sobolev spaces, equipped with anisotropic power weights. One of our main tools is a new Sobolev embedding result for weighted function spaces. In the main part of the paper we consider intersections of function spaces with mixed regularity. We prove mixed derivative embeddings with microscopic smoothing and obtain results on traces of intersected spaces. This provides an effective method to characterize trace spaces for evolution equations and boundary value problems. We illustrate this by means of a new result on maximal L^p - L^q -regularity for the linearized two-phase Stefan problem with Gibbs-Thomson correction.

1. INTRODUCTION

In recent years the L^p - L^q -maximal regularity approach to parabolic PDEs has attracted much attention. In the influential works [13, 30, 57] a new theory of maximal L^p -regularity was founded and many classes of examples are shown to have this property. Maximal regularity means that there is an isomorphism between the data and the solution of the linear problem in suitable function spaces. Having established such a sharp regularity result one can treat quasilinear problems by quite simple tools, like the contraction principle and the implicit function theorem (see [5, 12, 39] and references therein). Due to scaling invariance of PDEs one often requires $p \neq q$ for the underlying function space $L^p(L^q)$ (see e.g. [20] and [11, Section 3]), where p is the integrability in time and q is for the space variable.

In the L^p - L^q -approach to linear and quasilinear parabolic problems with nonhomogeneous boundary conditions it is essential to know the precise temporal and spatial trace spaces of the unknowns. In this way different types and scales of function spaces meet and come naturally into play. For example, in the L^p - L^q -approach to the heat equation one looks for strong solutions in the parabolic Sobolev space

$$H^{1,p}(\mathbb{R}_+; L^q(\mathbb{R}^d)) \cap L^p(\mathbb{R}_+; H^{2,q}(\mathbb{R}^d)),$$

whose temporal trace space at $t = 0$ is well-known to be the Besov space $B_{q,p}^{2-2/p}(\mathbb{R}^d)$. More recently, it turned out that the spatial trace space at the coordinate $x_d = 0$ is the intersection space

$$(1.1) \quad F_{p,q}^{1-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{2-1/q}(\mathbb{R}^{d-1})),$$

where $F_{p,q}^s$ denotes a Triebel-Lizorkin space. The spatial trace space (1.1) was obtained in [55, 56] for $q \leq p$ and more general cases were considered in [14, 27]. We conclude that the L^p - L^q -approach for already such a basic example as the heat equation with inhomogeneous boundary conditions involves three scales of function spaces.

In the case of free boundary problems or, more generally, for parabolic boundary value problems of relaxation type (see [15]), a second unknown is involved, which only lives on the boundary.

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For instance, for the transformed and linearized two-phase Stefan problem with Gibbs-Thomson correction, the optimal space for the boundary unknown is

$$(1.2) \quad F_{p,q}^{3/2-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap F_{p,q}^{1-1/(2q)}(\mathbb{R}_+; H^{2,q}(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{4-1/q}(\mathbb{R}^{d-1})),$$

see [15, 28] or Lemma 9.1 below. To treat the corresponding problem with nontrivial initial values one now has to determine the precise temporal trace space at $t = 0$ of this triple intersection space.

Moreover, if more than one boundary condition is involved, then mixed derivative embeddings are important to determine the optimal regularities of all boundary inhomogeneities, see [15, 16, 18, 28, 34, 35] and Section 7.

Stochastic parabolic equations and Volterra integral equations (see [37, 58, 59]) are other scenarios in which intersection spaces, even in an abstract form, come naturally into play. For an operator A with a bounded H^∞ -calculus on a space $X = L^q$ with $q \geq 2$ it is shown in [37] that the pathwise optimal regularity in the context of stochastic maximal L^p -regularity is

$$H^{s,p}(\mathbb{R}_+; X) \cap L^p(\mathbb{R}_+; D(A^{1/2-s})), \quad s \in [0, 1/2).$$

In many situations, e.g., when boundary conditions are involved, the fractional power domain $D(A^{1/2-s})$ is only a closed subspace of a function space as above. This is our motivation to study intersection spaces in an abstract form.

In a next step it is natural to introduce temporal power weights $w(t) = t^\gamma$ for the intersection spaces. Indeed, in many cases maximal regularity properties of parabolic problems are independent of the weight (see [35, 41]). Moreover, the weights yield flexibility for the initial regularity and thus a scale of phase spaces where the solution semiflow acts. This can be used to show a smoothing effect of the parabolic equation and compactness of the semiflow, which is an important property for the investigation of the long-time behavior of solutions (see [41, Remark 3.3] for a discussion).

In this article we can to a large extent overcome the problem of mixed regularity scales and study trace spaces and mixed derivative embeddings for a general class of intersection spaces on the line with power weights. Our results allow to characterize the regularity of the initial values in the temporally weighted L^p - L^q -approach to parabolic problems with general boundary conditions, as treated in [15, 28] (see also Remark 9.4). We demonstrate the scope of our methods by proving maximal L^p - L^q -regularity for the fully inhomogeneous linearized and transformed two-phase Stefan problem with Gibbs-Thomson correction.

As it turns out, the weights are not only useful for the applications. Sobolev embeddings for weighted Triebel-Lizorkin spaces $F_{p,q}^s$, being independent of the microscopic parameter q (see Section 3), are in fact the main technical tool in our investigations.

Let us describe the results in more detail. We start with the trace spaces at the hyperplane $\{(x', 0) : x' \in \mathbb{R}^{d-1}\}$ of the power weighted vector-valued function spaces

$$(1.3) \quad W^{k,p}(\mathbb{R}^d, w; X), \quad H^{s,p}(\mathbb{R}^d, w; X), \quad B_{p,q}^s(\mathbb{R}^d, w; X), \quad F_{p,q}^s(\mathbb{R}^d, w; X),$$

which denote Sobolev, Bessel-potential, Besov and Triebel-Lizorkin spaces (see Section 2.1 for the definitions). The next theorem is our main result. The proof will be given below in Proposition 4.4, Theorem 4.7 and Corollary 4.8.

Theorem 1.1. *Let X be a Banach space, $p \in (1, \infty)$, $q \in [1, \infty]$, $w(x', t) = |t|^\gamma$ with $\gamma > -1$ and $s > \frac{1+\gamma}{p}$. Then for the trace operator tr at $\{(x', 0) : x' \in \mathbb{R}^{d-1}\}$ it holds*

$$\text{tr}(B_{p,q}^s(\mathbb{R}^d, w; X)) = B_{p,q}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X), \quad \text{tr}(F_{p,q}^s(\mathbb{R}^d, w; X)) = B_{p,p}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X).$$

Assume in addition that $\gamma < p - 1$ and $m \in \mathbb{N}$. Then

$$\text{tr}(W^{m,p}(\mathbb{R}^d, w; X)) = B_{p,p}^{m-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X), \quad \text{tr}(H^{s,p}(\mathbb{R}^d, w; X)) = B_{p,p}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X).$$

Here the notation $\text{tr}(\mathcal{A}(\mathbb{R}^d, w; X)) = \mathcal{B}(\mathbb{R}^{d-1}; X)$ means that $\text{tr} : \mathcal{A}(\mathbb{R}^d, w; X) \rightarrow \mathcal{B}(\mathbb{R}^{d-1}; X)$ is continuous and surjective, and that it has a continuous right-inverse. The precise definition of the trace operator tr is given in Section 4.1.

Remark 1.2.

- (i) The weight $w(x', t) = |t|^\gamma$ can be seen as a power of the distant function to the hyperplane $\{(x', 0) : x' \in \mathbb{R}^{d-1}\}$ where the trace is taken. As a result, the exponent γ appears in the trace regularity $s - \frac{1+\gamma}{p}$. Of course, for $\gamma = 0$ one recovers the unweighted case. The situation of the theorem serves as a model case for spaces $F_{p,q}^s(\Omega, w; X)$, where $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain and $w(x) = \text{dist}(x, \partial\Omega)^\gamma$, see [29, 33].
- (ii) In the unweighted case $\gamma = 0$, the trace spaces of (1.3) are well-known at least for $X = \mathbb{C}$, see [45, 49, 50] and the references therein. Recently, also the general vector-valued case was investigated and the trace spaces were characterized in [42, 44]. Results for radial weights $w(x) = |x|^\gamma$ are obtained for scalar Sobolev spaces in [1], and in the general case for (1.3) in [24]. In the radial case the weight exponent does not appear in the trace regularity. Traces of Sobolev-Slobodetskii and Bessel-potential spaces with weights equal to a power of the distance to the boundary are intensively studied, see the references given in [50, Sections 2.9.2, 3.10.1]. In particular, in [22, Théorème 7.1] a partial result for Besov spaces is obtained.
- (iii) The additional condition $\gamma \in (-1, p-1)$ for the W - and H -spaces holds if and only if w belongs to the Muckenhoupt class A_p (see Section 2.1).

To determine the trace space of a Besov space in Theorem 1.1 we follow the arguments of [42] for the unweighted case. The result is in fact a consequence of the description of the B -spaces by the decomposition method and elementary Fourier analytic L^p -estimates for the trace of functions with compact Fourier support.

Our approach to the trace space of an F -space is surprisingly simple and is based on Sobolev embeddings and the trace space of a Besov space. In Section 3 we derive the embeddings for weighted spaces with anisotropic power weights are derived from our results in [36], where radial weights were considered. The trace space of a W - and H -space follows as in [42] from a sandwich argument, based on the embedding

$$(1.4) \quad F_{p,1}^s(\mathbb{R}^d, w; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X) \hookrightarrow F_{p,\infty}^s(\mathbb{R}^d, w; X),$$

and analogously with H replaced by W (see Section 2.1). Here the second embedding is valid only if w satisfies a local A_p -condition, which results in $\gamma < p-1$. The first embedding is true for all $w \in A_\infty$. We note at this point that $H^{s,p}(\mathbb{R}^d; X) = F_{p,2}^s(\mathbb{R}^d; X)$, i.e., the Littlewood-Paley decomposition for $L^p(\mathbb{R}^d; X)$, is valid if and only if X can be renormed as a Hilbert space.

It is known that the trace problem is intimately related to Hardy's inequality. As another consequence of the weighted Sobolev embeddings with fixed p we provide in Theorem 3.5 a microscopic improvement of Hardy's inequality in the target space. This is another hint for the canonical use of weights and Sobolev embeddings for trace considerations. The interpretation of Hardy's inequality as a Sobolev inequality with weights is well-known, see [51, Section 16.4].

The second and main part of our paper is on intersections of spaces with mixed regularity (e.g. $H^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A^\alpha))$). To investigate such spaces we first show an operator-valued Fourier multiplier theorem for B - and F -spaces with A_∞ -weights and give a characterization of these spaces in terms of differences, see Sections 5 and 6. Then we use the multiplier theorem to prove mixed derivative embeddings for abstract intersection spaces in Theorem 7.1. For B - and H -spaces, embeddings of this type are well-known and widely used in the context of boundary value problems with inhomogeneous symbols, see [15, 16, 18, 28, 34, 35].

In the case when the inner spaces are F -spaces, we prove a new smoothing effect with respect to the microscopic parameter. A special case of Theorem 7.3 is the following.

Theorem 1.3. *Let $d, m \in \mathbb{N}$, $p, p_0 \in (1, \infty)$, $q \in [1, \infty]$, $s_1, s_2 \in \mathbb{R}$, $\alpha_1, \alpha_2 > 0$ and $\theta \in (0, 1)$. Let $r_1 = s_1 + \theta\alpha_1$ and $r_2 = s_2 + (1-\theta)\alpha_2$. Then*

$$F_{p,q}^{s_1+\alpha_1}(\mathbb{R}^d; F_{p_0,\infty}^{s_2}(\mathbb{R}^m)) \cap F_{p,q}^{s_1}(\mathbb{R}^d; F_{p_0,\infty}^{s_2+\alpha_2}(\mathbb{R}^m)) \hookrightarrow F_{p,q}^{r_1}(\mathbb{R}^d; F_{p_0,1}^{r_2}(\mathbb{R}^m)),$$

The results remains true if one replaces F by B in the outer scale. In view of the monotonic properties of the F -spaces (2.7) and (1.4), the essential point of this result is the fact that on the

left-hand side in the inner scale one has the “large” spaces $F_{p_0, \infty}^{s_2}$ and $F_{p_0, \infty}^{s_2 + \alpha_2}$ and on the right-hand side the “small” spaces $F_{p_0, 1}^{r_2}$, hence there is a microscopic improvement. Using (1.4), the embeddings apply in particular to the intersection spaces of type (1.2). An analogous microscopic improvement for the outer regularity scale does not hold, see Proposition 7.5.

Our trace result for abstract weighted intersection spaces on the line is as follows. For the notion of sectorial operators and the real interpolation spaces $D_A(\theta, p)$ we refer to Section 2.2.

Theorem 1.4. *Let $p \in (1, \infty)$ and $w(t) = |t|^\gamma$ with $\gamma \in (-1, p-1)$, suppose that $s \in \mathbb{R}$ and $\alpha > 0$ satisfy $s < \frac{1+\gamma}{p} < s + \alpha$, and let A be a sectorial operator on a Banach space X with spectral angle $\phi_A < \min\{\frac{\pi}{2}, \frac{\pi}{\alpha}\}$ and $r \geq 0$. Let $\theta = r + s + \alpha - \frac{1+\gamma}{p}$. Then for the trace operator tr_0 at $t = 0$ it holds*

$$\begin{aligned} \text{tr}_0(F_{p,q}^{s+\alpha}(\mathbb{R}, w; D(A^r)) \cap F_{p,q}^s(\mathbb{R}, w; D(A^{r+\alpha}))) &= D_A(\theta, p), \quad q \in [1, \infty], \\ \text{tr}_0(B_{p,p}^{s+\alpha}(\mathbb{R}, w; D(A^r)) \cap B_{p,p}^s(\mathbb{R}, w; D(A^{r+\alpha}))) &= D_A(\theta, p), \\ \text{tr}_0(H^{s+\alpha,p}(\mathbb{R}, w; D(A^r)) \cap H^{s,p}(\mathbb{R}, w; D(A^{r+\alpha}))) &= D_A(\theta, p), \\ \text{tr}_0(W^{s+\alpha,p}(\mathbb{R}, w; D(A^r)) \cap H^{s,p}(\mathbb{R}, w; D(A^{r+\alpha}))) &= D_A(\theta, p). \end{aligned}$$

Remark 1.5.

- (i) If $s > \frac{1+\gamma}{p} - 1$, then the right-inverse of tr_0 can be chosen such that it only depends on A . In the other cases the extension operator in our proof depends on s , α and A . As shown in Lemma 8.1, tr_0 maps continuously to $D_A(\theta, p)$ for all $\gamma > -1$.
- (ii) The essential point is the independence of the trace space of the F -intersection spaces of the microscopic parameter $q \in [1, \infty]$. It follows from a sandwich argument that the trace space of spaces with mixed regularity inside the F -scale is also equal to $D_A(\theta, p)$. More precisely, let $\mathcal{A}_p^{s+\alpha}$ and \mathcal{B}_p^s be function spaces such that

$$F_{p,1}^{s+\alpha} \hookrightarrow \mathcal{A}_p^{s+\alpha} \hookrightarrow F_{p,\infty}^{s+\alpha}, \quad F_{p,1}^s \hookrightarrow \mathcal{B}_p^s \hookrightarrow F_{p,\infty}^s,$$

see e.g. (1.4). Then the theorem shows that

$$\text{tr}_0(\mathcal{A}_p^{s+\alpha}(\mathbb{R}, w; D(A^r)) \cap \mathcal{B}_p^s(\mathbb{R}, w; D(A^{r+\alpha}))) = D_A(\theta, p).$$

In special situations, real interpolation allows to replace e.g. $D(A^{r+\alpha})$ by $D_A(r + \alpha, p)$, see the proof of [34, Theorem 4.2]. Moreover, in combination with Theorem 1.3 one obtains independence in the outer and the inner regularity scale in the context of traces.

- (iii) As mentioned before, the intersection spaces and their variants arise in the maximal L^p - L^q -regularity approach to deterministic and stochastic parabolic evolution equations, see [14, 15, 16, 18, 28, 35, 37, 41, 55, 58, 59].
- (iv) The theorem applies in particular to any fractional power of $-\Delta$ on

$$X \in \{H^{r,q}(\mathbb{R}^d), B_{q,\sigma}^r(\mathbb{R}^d), F_{q,\sigma}^r(\mathbb{R}^d) : r \in \mathbb{R}, q \in (1, \infty), \sigma \in [1, \infty]\}.$$

Since the spectral angle of $-\Delta$ is equal to zero, $\alpha > 0$ above may be arbitrary large.

- (v) The result is a generalization of [34, Theorem 4.2] in the weighted case, and of [16, Theorem 4.5] and [59, Theorem 3.6] (see also [58, Theorem 3.1.4]) in the unweighted case. In these works only the H - and B -spaces were considered. The continuity of tr_0 was proved in [59, Theorem 3.6] under the assumption that X is a UMD Banach space and that A is \mathcal{R} -sectorial with \mathcal{R} -angle not larger than $\frac{\pi}{\alpha}$. The reason for these stronger assumptions is that the proof in [59] relies on the operator-valued Fourier multiplier result due to [57]. Moreover, the proof uses a result on complex interpolation of H -spaces with Dirichlet boundary conditions (see [46], and [6] for the vector-valued case), which is not trivial to extend to weighted case. The condition $\phi_A < \frac{\pi}{2}$ is not assumed in [59]. Also for Theorem 1.4 it should not be essential.
- (vi) The traces of anisotropic Besov and Triebel-Lizorkin spaces, not necessarily of intersection type, are studied in [6, 8, 27]. However, there are only partial results on how the spaces considered there are related to the intersection spaces when X and $D(A)$ are function spaces over \mathbb{R}^d as above, see [6, Sections 3.6-3.8], [27, Section 5] and [28, Section 5.2]. In particular, the case of an operator with boundary conditions on a domain is not included there.

- (vii) It should be possible to generalize Theorem 1.4 to dimensions $d \geq 2$. Given the situation of the theorem, using a Fubini argument and the methods of Section 8 one can see that the trace tr at $\{(x', 0) : x' \in \mathbb{R}^{d-1}\}$ maps

$$F_{p,q}^{s+\alpha}(\mathbb{R}^d, w; D(A^r)) \cap F_{p,q}^s(\mathbb{R}^d, w; D(A^{r+\alpha}))$$

continuously into

$$B_{p,p}^{s+\alpha-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; D(A^r)) \cap L^p(\mathbb{R}^{d-1}; D_A(\theta, p)).$$

We expect that this is indeed the trace space. It seems elaborate to construct a right-inverse.

Theorem 1.4 is shown in Section 8. The proof of the continuity of tr_0 is based on a semigroup representation of the real interpolation spaces $D_A(\theta, p)$, Hardy's inequality, the equivalent norm in terms of differences for the F -spaces and weighted Sobolev embeddings (see Lemma 8.1). The right-inverse for tr_0 is essentially the semigroup $e^{-\cdot A}$, combined with an extension operator to \mathbb{R} . The proof of the desired mapping properties in Lemma 8.3 is once more based on Sobolev embeddings.

In the final Section 9 we apply the general results and prove maximal L^p - L^q -regularity in the parameter range $\frac{2p}{p+1} < q < 2p$ for the fully inhomogeneous linearized two-phase Stefan problem with Gibbs-Thomson correction (see [18] and the references therein). The corresponding original problem is a free boundary problem which can be used to model the melting of ice. Based on the results of [28] for trivial initial values, our main contribution is here to determine the temporal trace space of (1.2). The case $p = q$ was considered in [16] for the one phase problem, and in [18] for the two-phase problem.

This paper is organized as follows.

- Section 2: preliminaries on weighted function spaces.
- Section 3: Sobolev embeddings theorem with an anisotropic power weight.
- Section 4: trace space of weighted function spaces (Theorem 1.1).
- Section 5: operator-valued Fourier multiplier theorem for weighted F and B -spaces.
- Section 6: equivalent norms in terms of differences for weighted F and B -spaces.
- Section 7: mixed derivative embeddings for spaces of intersection type (Theorem 1.3).
- Section 8: traces of intersection spaces (Theorem 1.4).
- Section 9: linearized two-phase Stefan problem with Gibbs-Thomson correction.

2. PRELIMINARIES

2.1. Function spaces and elementary embeddings. We briefly recall the definitions and embedding properties of the weighted vector-valued function spaces. For details and references we refer to [36, Sections 2 and 3].

For $p \in (1, \infty)$ the Muckenhoupt class of weights is denoted by A_p , and $A_\infty = \bigcup_{p>1} A_p$. In the present work we mainly consider power weights w on \mathbb{R}^d , $d \geq 1$, of the form

$$w(x', t) = |t|^\gamma, \quad x = (x', t) \in \mathbb{R}^d, \quad x' \in \mathbb{R}^{d-1}, \quad t \in \mathbb{R}.$$

Here we have $w \in A_p$ if and only if $\gamma \in (-1, p-1)$ (see [25, Example 1.5]).

For a Banach space X , $p \in (1, \infty)$ and $w \in A_\infty$ the norm of $L^p(\mathbb{R}^d, w; X)$ is given by

$$\|f\|_{L^p(\mathbb{R}^d, w; X)} := \left(\int_{\mathbb{R}^d} \|f(x)\|_X^p w(x) dx \right)^{1/p}.$$

One further defines $L^\infty(\mathbb{R}^d, w; X) := L^\infty(\mathbb{R}^d; X)$. Let $\mathcal{S}(\mathbb{R}^d; X)$ be the Schwartz class of X -valued, smooth rapidly decreasing functions on \mathbb{R}^d , and let $\mathcal{S}'(\mathbb{R}^d; X) = \mathcal{L}(\mathcal{S}(\mathbb{R}^d); X)$ be the space of X -valued tempered distributions. The Fourier transform of a distribution f is denoted by $\mathcal{F}f$ or \widehat{f} .

Let $\Phi(\mathbb{R}^d)$ be the set of all sequences $(\varphi_k)_{k \geq 0} \subseteq \mathcal{S}(\mathbb{R}^d)$ such that

$$(2.1) \quad \widehat{\varphi}_0 = \widehat{\varphi}, \quad \widehat{\varphi}_1(\xi) = \widehat{\varphi}(\xi/2) - \widehat{\varphi}(\xi), \quad \widehat{\varphi}_k(\xi) = \widehat{\varphi}_1(2^{-k+1}\xi), \quad k \geq 2, \quad \xi \in \mathbb{R}^d,$$

where the Fourier transform of the generating function φ satisfies

$$(2.2) \quad 0 \leq \widehat{\varphi}(\xi) \leq 1, \quad \xi \in \mathbb{R}^d, \quad \widehat{\varphi}(\xi) = 1 \text{ if } |\xi| \leq 1, \quad \widehat{\varphi}(\xi) = 0 \text{ if } |\xi| \geq \frac{3}{2}.$$

For $(\varphi_k)_{k \geq 0} \in \Phi(\mathbb{R}^d)$ and $f \in \mathcal{S}'(\mathbb{R}^d; X)$ we set

$$S_k f = \varphi_k * f = \mathcal{F}^{-1}(\widehat{\varphi}_k \widehat{f}).$$

As a consequence of the boundedness of the Hardy-Littlewood maximal function on $L^p(\mathbb{R}^d, w)$, for $w \in A_p$ we have (see e.g. [36, Lemma 2.3])

$$(2.3) \quad \sup_{k \geq 0} \|S_k f\|_{L^p(\mathbb{R}^d, w; X)} \leq C \|f\|_{L^p(\mathbb{R}^d, w; X)}.$$

Now fix a Banach space X , $p \in (1, \infty)$, $q \in [1, \infty]$ and $w \in A_\infty$. Then for $f \in \mathcal{S}'(\mathbb{R}^d; X)$ we set

$$\begin{aligned} \|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)} &:= \left\| (2^{ks} S_k f)_{k \geq 0} \right\|_{\ell^q(L^p(\mathbb{R}^d, w; X))}, \\ \|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)} &:= \left\| (2^{ks} S_k f)_{k \geq 0} \right\|_{L^p(\mathbb{R}^d, w; \ell^q(X))}, \\ \|f\|_{H^{s,p}(\mathbb{R}^d, w; X)} &:= \|\mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2} \widehat{f}]\|_{L^p(\mathbb{R}^d, w; X)}. \end{aligned}$$

These norms define the Besov space $B_{p,q}^s(\mathbb{R}^d, w; X)$, the Triebel-Lizorkin space $F_{p,q}^s(\mathbb{R}^d, w; X)$, and the Bessel-potential space $H^{s,p}(\mathbb{R}^d, w; X)$, respectively, which are all Banach spaces. Any other $(\psi_k)_{k \geq 0} \in \Phi$ leads to an equivalent norm on the B - and F -spaces. If $s \in \mathbb{R}_+ \setminus \mathbb{N}$, then we set

$$W^{s,p}(\mathbb{R}^d, w; X) := B_{p,p}^s(\mathbb{R}^d, w; X),$$

and for $m \in \mathbb{N}_0$,

$$\|f\|_{W^{m,p}(\mathbb{R}^d, w; X)} := \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\mathbb{R}^d, w; X)},$$

where the derivatives D^α are taken in a distributional sense. These norms define the Slobodetskii and the Sobolev spaces, respectively. Note that $L^p(\mathbb{R}^d, w; X) = H^{0,p}(\mathbb{R}^d, w; X) = W^{0,p}(\mathbb{R}^d, w; X)$.

Each of the above spaces embeds continuously into $\mathcal{S}'(\mathbb{R}^d; X)$. Conversely, $\mathcal{S}(\mathbb{R}^d; X)$ embeds continuously into each of the above spaces, where this is a dense embedding if $p, q < \infty$.

Let a Banach space E be continuously embedded into $\mathcal{S}'(\mathbb{R}^d; X)$. The space E is said to have the *Fatou property* if for all $(f_n)_{n \geq 0} \subseteq E$ it holds

$$\lim_{n \rightarrow \infty} f_n = f \text{ in } \mathcal{S}'(\mathbb{R}^d; X), \quad \liminf_{n \rightarrow \infty} \|f_n\|_E < \infty \implies f \in E, \quad \|f\|_E \leq \liminf_{n \rightarrow \infty} \|f_n\|_E.$$

Proposition 2.1. *Let X be a Banach space, $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty]$ and $w \in A_\infty$. Then each of the spaces $B_{p,q}^s(\mathbb{R}^d, w; X)$ and $F_{p,q}^s(\mathbb{R}^d, w; X)$ has the Fatou property.*

Such properties were used in [19] for the first time and independently in [9]. For a slightly different formulation as ours and a proof we refer to [42, Proposition 2.18] and [44, Proposition 4]. The Fatou property is useful as a substitute for the density of $\mathcal{S}(\mathbb{R}^d; X)$ in the B - and F -spaces in the important case when $q = \infty$.

There are elementary embeddings between the function spaces, see [36, Propositions 3.11 and 3.12]. For $s \in \mathbb{R}$ and $m \in \mathbb{N}_0$ we shall make particular use of

$$(2.4) \quad F_{p,1}^s(\mathbb{R}^d, w; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X) \hookrightarrow F_{p,\infty}^s(\mathbb{R}^d, w; X),$$

$$(2.5) \quad F_{p,1}^m(\mathbb{R}^d, w; X) \hookrightarrow W^{m,p}(\mathbb{R}^d, w; X) \hookrightarrow F_{p,\infty}^m(\mathbb{R}^d, w; X).$$

The above embeddings into $F_{p,\infty}^s$ and $F_{p,\infty}^m$ are valid for $w \in A_p$. In [36, Remark 3.13] it is shown that a local A_p -condition is necessary for (2.4) and (2.5) to hold. For power weights $w(x', t) = |t|^\gamma$ as above this condition is equivalent to the usual A_p -condition. The above embeddings for $F_{p,1}^s(\mathbb{R}^d, w; X)$ are valid for all $w \in A_\infty$. For all $q \in [1, \infty]$, $s \in \mathbb{R}$ and $p \in (1, \infty)$ one further has

$$(2.6) \quad B_{p,\min\{p,q\}}^s(\mathbb{R}^d, w; X) \hookrightarrow F_{p,q}^s(\mathbb{R}^d, w; X) \hookrightarrow B_{p,\max\{p,q\}}^s(\mathbb{R}^d, w; X),$$

and if $1 \leq q_0 \leq q_1 \leq \infty$, then

$$(2.7) \quad B_{p,q_0}^s(\mathbb{R}^d, w; X) \hookrightarrow B_{p,q_1}^s(\mathbb{R}^d, w; X), \quad F_{p,q_0}^s(\mathbb{R}^d, w; X) \hookrightarrow F_{p,q_1}^s(\mathbb{R}^d, w; X).$$

2.2. Sectorial operators and real interpolation. Below we only recall some standard definitions and results on sectorial operators. For a detailed exposition we refer to [3, 13, 32, 50]. Let A be a densely defined operator on a Banach space X . Then A is called *sectorial* on $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$, $\theta \in (0, \pi)$, if Σ_θ is contained in the resolvent set of $-A$ and if there is a constant $C > 0$ such that $\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C$. The infimum of all θ with this property is denoted by ϕ_A and is called the *spectral angle* of A . The operator $-A$ generates an analytic C_0 -semigroup if and only if it is sectorial with $\phi_A < \frac{\pi}{2}$.

For $\alpha \in \mathbb{R}$ the *fractional power* A^α of a sectorial operator A is defined by the extended Dunford calculus as in [13, Section 2]. If $|\alpha| < \frac{\pi}{\phi_A}$, then A^α is also sectorial with $\phi_{A^\alpha} \leq |\alpha|\phi_A$ (see [13, Theorem 2.3]).

For $s \in (0, 1)$ and $p \in [1, \infty]$ the real interpolation functor is denoted by $(\cdot, \cdot)_{s,p}$. For an invertible sectorial operator A we write

$$D_A(k + \theta, p) = \{x \in D(A^k) : A^k x \in (X, D(A))_{\theta,p}\}, \quad k \in \mathbb{N}_0, \quad \theta \in (0, 1), \quad p \in [1, \infty].$$

For our purposes it is further convenient to define

$$D_A(k, p) = \{x \in X : A^{1/2} x \in D_A(k - 1/2, p)\}, \quad k \in \mathbb{N}.$$

Here $1/2$ may be replaced by any other $\theta \in (0, 1)$. With this notation, the reiteration theorem and [50, Theorem 1.15.2] show that for all $\theta, \alpha > 0$ the operator A^α is an isomorphism $D_A(\theta + \alpha, p) \rightarrow D_A(\theta, p)$ and $D(A^{\theta+\alpha}) \rightarrow D(A^\theta)$, respectively. If $-A$ is invertible and generates an analytic C_0 -semigroup, then it follows from [50, Theorem 1.14.5] that

$$(2.8) \quad x \mapsto \left(\int_0^\infty \sigma^{p(1-\theta)} \|Ae^{-\sigma A} x\|_X^p \frac{d\sigma}{\sigma} \right)^{1/p}$$

defines an equivalent norm on $D_A(\theta, p)$ for $\theta \in (0, 1)$.

3. SOBOLEV EMBEDDINGS FOR ANISOTROPIC POWER WEIGHTS

In this section we prove Sobolev embeddings for function spaces with anisotropic weights. In the case of Triebel-Lizorkin spaces these embeddings are independent of the microscopic parameter $q \in [1, \infty]$. In [36, Theorems 1.1 and 1.2] we have characterized such embeddings results for spaces with *radial* power weights in terms of the parameters. The flexibility in both the microscopic parameter $q \in [1, \infty]$ and the weights, combined with elementary embeddings (2.4) and (2.5), are key elements in our investigations of trace spaces.

From [36, Theorems 1.1 and 1.2] one obtains the following sufficient conditions for Sobolev embeddings with radial weights.

Theorem 3.1. *Let X be Banach space, $1 < p_0 \leq p_1 < \infty$, $q_0, q_1 \in [1, \infty]$, $s_0 > s_1$ and $w_0(x) = |x|^{\gamma_0}$, $w_1(x) = |x|^{\gamma_1}$ with $\gamma_0, \gamma_1 > -d$. Suppose that*

$$\frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0} \quad \text{and} \quad s_0 - \frac{d + \gamma_0}{p_0} = s_1 - \frac{d + \gamma_1}{p_1}.$$

Then one has the continuous embedding

$$(3.1) \quad F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1; X).$$

Suppose in addition that $q_0 \leq q_1$. Then

$$(3.2) \quad B_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow B_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1; X).$$

Our arguments for the corresponding embeddings for anisotropic weights are based on the following inequality of Plancherel-Polya-Nikol'skij type, see [36, Proposition 4.1].

Lemma 3.2. *Let X be a Banach space and let $1 < p_0, p_1 \leq \infty$. Let $\gamma_0, \gamma_1 > -d$ and $w_0(x) = |x|^{\gamma_0}$ and $w_1(x) = |x|^{\gamma_1}$. Suppose*

$$\frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0} \quad \text{and} \quad \frac{d + \gamma_1}{p_1} < \frac{d + \gamma_0}{p_0}.$$

Let $R > 0$ and let $f : \mathbb{R}^d \rightarrow X$ be a function with $\text{supp } \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| < R\}$. Then there is a constant C , independent of f and R , such that

$$\|f\|_{L^{p_1}(\mathbb{R}^d, w_1; X)} \leq CR^\delta \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)},$$

where $\delta = \frac{d+\gamma_0}{p_0} - \frac{d+\gamma_1}{p_1} > 0$.

We prove the following complement of Theorem 3.1. In the scalar case $X = \mathbb{C}$, the embeddings for the Besov spaces and their optimality can be deduced from [25, Proposition 2.8].

Theorem 3.3. *Let X be a Banach space, $1 < p_0 \leq p_1 < \infty$, $q_0, q_1 \in [1, \infty]$, $s_0 > s_1$ and $w_0(x', t) = |t|^{\gamma_0}$, $w_1(x', t) = |t|^{\gamma_1}$ with $\gamma_0, \gamma_1 > -1$. Suppose that*

$$(3.3) \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0} \quad \text{and} \quad s_0 - \frac{d+\gamma_0}{p_0} = s_1 - \frac{d+\gamma_1}{p_1}.$$

Then one has the continuous embedding

$$(3.4) \quad F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1; X).$$

Suppose in addition that $q_0 \leq q_1$. Then

$$(3.5) \quad B_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow B_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1; X).$$

Proof. Step 1. We first show (3.5). By (2.7), it suffices to consider the case $q := q_0 = q_1$. We write $s_1 + \delta = s_0$, where $\delta = \delta' + \delta''$ with $\delta' = \frac{\gamma_0+1}{p_0} - \frac{\gamma_1+1}{p_1}$ and $\delta'' = \frac{d-1}{p_0} - \frac{d-1}{p_1}$. Let $f \in B_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X)$ and set $f_n = S_n f$.

Note that each f_n is a smooth bounded function on \mathbb{R}^d , and that $\text{supp } \widehat{f_n} \subseteq \{|\xi| \leq 3 \cdot 2^{n-1}\}$. Let \mathcal{F}_t be the Fourier transform with respect to $t \in \mathbb{R}$. It follows from Step 2 of the proof of [42, Theorem 4.9] that for each fixed $x \in \mathbb{R}^{d-1}$ the function $\mathcal{F}_t(f_n(x', \cdot))$ is supported in $\{|\lambda| < 3 \cdot 2^{n-1}\}$. We use Lemma 3.2 and Minkowski's inequality (or the triangle inequality for Bochner integrals in $\|\cdot\|_{L^{p_1/p_0}(\mathbb{R}, w_0)}$) to estimate

$$\begin{aligned} \|f_n\|_{L^{p_1}(\mathbb{R}^d, w_1; X)} &= \left(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \|f_n(x', t)\|^{p_1} |t|^{\gamma_1} dt dx' \right)^{1/p_1} \\ &\leq C 2^{\delta' n} \left(\int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \|f_n(x', t)\|^{p_0} |t|^{\gamma_0} dt \right)^{p_1/p_0} dx' \right)^{1/p_1} \\ &\leq C 2^{\delta' n} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} \|f_n(x', t)\|^{p_1} dx' \right)^{p_0/p_1} |t|^{\gamma_0} dt \right)^{1/p_0}. \end{aligned}$$

As above, for fixed $t \in \mathbb{R}$ the function $\mathcal{F}_{x'}(f_n(\cdot, t))$ is supported in $\{|\xi'| < 3 \cdot 2^{n-1}\}$. Again Lemma 3.2 gives

$$\begin{aligned} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} \|f_n(x', t)\|^{p_1} dx' \right)^{p_0/p_1} |t|^{\gamma_0} dt \right)^{1/p_0} &\leq C 2^{\delta'' n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \|f_n(x', t)\|^{p_0} dx' |t|^{\gamma_0} dt \right)^{1/p_0} \\ &= C 2^{\delta'' n} \|f_n\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}. \end{aligned}$$

Therefore

$$(3.6) \quad \|f_n\|_{L^{p_1}(\mathbb{R}^d, w_1; X)} \leq C 2^{\delta n} \|f_n\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}, \quad n \in \mathbb{N}_0,$$

where C does not depend on f and n . Using $s_1 + \delta = s_0$, we find that

$$\begin{aligned} \|f\|_{B_{p_1, q}^{s_1}(\mathbb{R}^d, w_1; X)} &= \|(2^{s_1 n} \|f_n\|_{L^{p_1}(\mathbb{R}^d, w_1; X)})_{n \geq 0}\|_{\ell^q} \\ &\leq C \|(2^{s_0 n} \|f_n\|_{L^{p_0}(\mathbb{R}^d, w_0; X)})_{n \geq 0}\|_{\ell^q} = C \|f\|_{B_{p_0, q}^{s_0}(\mathbb{R}^d, w_0; X)}. \end{aligned}$$

This proves (3.5).

Step 2. Using [36, Proposition 5.1], now the embedding (3.4) can be deduced from (3.5). In fact, literally in the same way as in the proof of [36, Theorem 1.2] we obtain (3.4) for all $f \in F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1; X) \cap F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X)$. The general case follows from a Fatou argument, which we sketch here for the convenience of the reader.

For $f \in F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w; X)$ the series $f_N = \sum_{n=0}^N f_n$ converges to f in $\mathcal{S}'(\mathbb{R}^d; X)$ as $N \rightarrow \infty$, and it holds $f_N \in F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w; X) \cap F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w; X)$ for each N by (3.6). Using [36, Proposition 2.4], we further have

$$\|f_N\|_{F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w; X)} \leq C \|f\|_{F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w; X)},$$

and therefore

$$\|f_N\|_{F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w; X)} \leq C \|f\|_{F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w; X)}.$$

Now Proposition 2.1 gives $f \in F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w; X)$ and

$$\|f\|_{F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w; X)} \leq \liminf_{N \rightarrow \infty} \|f_N\|_{F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w; X)} \leq C \|f\|_{F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w; X)},$$

which finishes the proof. \square

Remark 3.4.

- (i) In the Besov scale Theorem 3.1 has an extension to $p_0 > p_1$ (see [36, Theorem 1.1]).
- (ii) If $d = 1$, then Theorems 3.1 and 3.3 coincide.
- (iii) If $d \geq 2$, then Theorem 3.3 does not have an extension to $p_0 > p_1$. Indeed, let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a function with sufficiently small Fourier support and let f_r be a dilatation only in the x' -variable, i.e., $f_r(x', t) = f(rx', t)$ for $r > 0$. Then one can repeat the argument in [36, Proposition 4.7] to see that $\frac{d-1}{p_1} \leq \frac{d-1}{p_0}$, which implies $p_0 \leq p_1$.
- (iv) Theorem 3.3 is also a model case for spaces over a bounded smooth domain $\Omega \subset \mathbb{R}^d$ with weights $w_j(x) = \text{dist}(x, \partial\Omega)^{\gamma_j}$, see [29, 33].
- (v) If $\gamma_0 < p - 1$, then one can combine the result for the F -spaces with the embeddings (2.4) and (2.5) to obtain Sobolev embeddings for weighted H - and W -spaces.

As an application we give a very short proof and a generalization *in the target space* of a well-known Hardy inequality. Early versions of Hardy's inequality can be found in [22], [31, Proposition 2.3] and [51, Proposition 5.7] and references therein. The inequality seems to go back to certain inequalities for double integrals in [7].

Theorem 3.5. *Let X be a Banach space, $p \in (1, \infty)$, $q \in [1, \infty]$ and $w(x', t) = |t|^\gamma$ for $\gamma > -1$. Suppose that $0 < s < \frac{1+\gamma}{p}$ and $v(x', t) = |t|^{\gamma-ps}$. Then one has*

$$\|f\|_{L^p(\mathbb{R}^d, v; X)} \leq C \|f\|_{F_{p,1}^0(\mathbb{R}^d, v; X)} \leq C \|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)}, \quad f \in F_{p,q}^s(\mathbb{R}^d, w; X).$$

In particular, the estimate holds with $F_{p,q}^s(\mathbb{R}^d, w; X)$ replaced by $B_{p,p}^s(\mathbb{R}^d, w; X)$. If additionally $\gamma < p - 1$, then the estimate holds with $F_{p,q}^s(\mathbb{R}^d, w; X)$ replaced by $H_p^s(\mathbb{R}^d, w; X)$.

Proof. Let $w_1(x', t) = |t|^{\gamma-ps}$. Then by assumption $\gamma - ps > -1$ and hence $w_1 \in A_\infty$. Since $f = \sum_{n \geq 0} S_n f$, the triangle inequality and Theorem 3.3 yield

$$\|f\|_{L^p(\mathbb{R}^d, dx' |t|^{\gamma-ps} dt; X)} \leq \left\| \sum_{n \geq 0} \|S_n f\|_X \right\|_{L^p(w_1)} = \|f\|_{F_{p,1}^0(\mathbb{R}^d, w_1; X)} \leq C \|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)}.$$

The final assertion follows from (2.4) and $w \in A_p$ if $-1 < \gamma < p - 1$. \square

Remark 3.6.

- (i) The same result as in Theorem 3.5 holds for the radial weights $w(x) = |x|^\gamma$ and $v(x) = |x|^{\gamma-ps}$, where $0 < s < \frac{d+\gamma}{p}$. Indeed, this is a special case of Theorem 3.1. This should be compared to [51, Section II.16] and the references given therein. In particular, it is explained that Hardy's inequalities may be seen as Sobolev embeddings for weighted spaces. Our result seems to be a strengthening of this and shows the usefulness of Triebel-Lizorkin spaces for this matter.
- (ii) The usual formulation of Hardy's inequality in terms of increments can be derived from Theorem 3.5 and Proposition 6.1 below.
- (iii) Using homogenous versions of Theorems 3.1 and 3.3 one can also derive inequalities of Hardy type for homogenous norms.

4. TRACES OF FUNCTION SPACES WITH ANISOTROPIC POWER WEIGHTS

In this section we determine the trace spaces of F -, B -, H - and W -spaces with anisotropic weights.

4.1. Definition of the trace. In the important case $q = \infty$ the Schwartz functions are not dense in the B - and F -spaces, and thus one cannot define the trace on these spaces as an extension of the trace in the classical sense. For a unified approach one has to consider a more general definition of a trace. As in [42] we employ a concept due to Nikol'skij [38]. Let $f : \mathbb{R}^d \rightarrow X$ be strongly measurable. Then $g : \mathbb{R}^{d-1} \rightarrow X$ is called the trace of f on $\mathbb{R}^{d-1} \times \{0\}$, if there are $\tilde{f} : \mathbb{R}^d \rightarrow X$, $p \in [1, \infty]$ and $\delta > 0$ such that

- (i) $\tilde{f} = f$ a.e. with respect to the Lebesgue measure on \mathbb{R}^d ;
- (ii) $\tilde{f}(\cdot, t) \in L^p(\mathbb{R}^{d-1}; X)$ for $|t| < \delta$;
- (iii) $\tilde{f}(\cdot, 0) = g$ a.e. with respect to the Lebesgue measure on \mathbb{R}^{d-1} ;
- (iv) $\lim_{t \rightarrow 0} \|\tilde{f}(\cdot, t) - g\|_{L^p(\mathbb{R}^{d-1}; X)} = 0$.

This definition is independent of \tilde{f} , p and δ (see [42, Remark 4.2]), and it coincides with the restriction of f to $\{(x', 0) : x' \in \mathbb{R}^{d-1}\}$ if f is continuous on $\mathbb{R}^{d-1} \times (-\delta, \delta)$. If the trace exists in the above sense, then we write

$$\text{tr} f := g,$$

and obtain in this way a linear operator tr .

4.2. The trace space of a Besov space. The results and proofs in this subsection are completely analogous to the results for unweighted vector-valued Besov spaces obtained in [42]. Since the weight disappears for $p = \infty$, in the proofs below we only consider the case $p < \infty$.

We first generalize [42, Lemma 4.5] to the weighted setting.

Lemma 4.1. *Let X be a Banach space, $p \in (1, \infty)$ and $w(x', t) = |t|^\gamma$ with $\gamma > -1$. Let $f \in L^p(\mathbb{R}^d; X)$ be such that $\text{supp } \hat{f} \subseteq \{|\xi| \leq R\}$ for some $R > 0$. Then there is a constant C , independent of f and R , such that for all $t \in \mathbb{R}$ we have*

$$\|f(\cdot, t)\|_{L^p(\mathbb{R}^{d-1}; X)} \leq CR^{\frac{1+\gamma}{p}} \|f\|_{L^p(\mathbb{R}^d, w; X)}.$$

Proof. By a scaling and a translation argument it suffices to consider the case $R = 1$ and $t = 0$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be such that $\psi \equiv 1$ on B_1 . Then for all $x' \in \mathbb{R}^{d-1}$ we have

$$f(x', 0) = \mathcal{F}^{-1}(\hat{\psi} \hat{f})(x', 0) = \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} f(x' - y', -t) |t|^{\frac{\gamma}{p}} \psi(y', t) |t|^{-\frac{\gamma}{p}} dt \right) dy'.$$

Thus Hölder's inequality gives, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\gamma' = -\frac{\gamma}{p-1}$,

$$\|f(x', 0)\| \leq \int_{\mathbb{R}^{d-1}} \|f(x' - y', \cdot)\|_{L^p(\mathbb{R}, |\cdot|^\gamma; X)} \|\psi(y', \cdot)\|_{L^{p'}(\mathbb{R}, |\cdot|^{\gamma'}; X)} dy'.$$

Taking the L^p -norm with respect to x' and using Minkowski's inequality, we obtain

$$\|f(x', 0)\|_{L^p(\mathbb{R}^{d-1}; X)} \leq \|f\|_{L^p(\mathbb{R}^d, w; X)} \int_{\mathbb{R}^{d-1}} \|\psi(y', \cdot)\|_{L^{p'}(\mathbb{R}, |\cdot|^{\gamma'}; X)} dy'.$$

Since $\gamma' > -1$, the second factor is finite. \square

The next result is analogous to [42, Proposition 4.4].

Proposition 4.2. *Let X be a Banach space, $p \in (1, \infty)$ and $w(x', t) = |t|^\gamma$ with $\gamma > -1$. Then the trace of $f \in B_{p,1}^{\frac{1+\gamma}{p}}(\mathbb{R}^d, w; X)$ exists, and for any $(\varphi_n)_{n \geq 0} \subseteq \Phi(\mathbb{R}^d)$ it holds*

$$\text{tr} f = \sum_{n=0}^{\infty} \varphi_n * f(\cdot, 0),$$

where the convergence is in $L^p(\mathbb{R}^{d-1}; X)$.

Proof. As in [42], for $f \in B_{p,1}^{\frac{1+\gamma}{p}}(\mathbb{R}^d, w; X)$ and $(\varphi_n)_{n \geq 0} \in \Phi(\mathbb{R}^d)$ we let $\tilde{f} = \sum_{n=0}^{\infty} S_n f$ and verify that it satisfies the requirements from the above definition. Note that $\sum_{n=0}^{\infty} \|S_n f\|_{L^p(\mathbb{R}^d, w; X)} < \infty$, since $f \in B_{p,1}^{\frac{1+\gamma}{p}}(\mathbb{R}^d, w; X)$

The series converges to f in $L^p(\mathbb{R}^d, w; X)$, and thus $\tilde{f} = f$ a.e. on \mathbb{R}^d . By Lemma 4.1 and $\widehat{S_n f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leq 3 \cdot 2^n\}$, for $t \in \mathbb{R}$ we have

$$\begin{aligned} \|\tilde{f}(\cdot, t)\|_{L^p(\mathbb{R}^{d-1}; X)} &\leq \|(\|S_n f(\cdot, t)\|_{L^p(\mathbb{R}^{d-1}; X)})_{n \geq 0}\|_{\ell^1} \\ &\leq C \|(2^{\frac{1+\gamma}{p}n} \|S_n f\|_{L^p(\mathbb{R}^d, w; X)})_{n \geq 0}\|_{\ell^1} = C \|f\|_{B_{p,1}^{\frac{1+\gamma}{p}}(\mathbb{R}^d, w; X)}. \end{aligned}$$

For the last condition we use Lemma 4.1 and the pointwise estimate from [42, Lemma 2.3] to obtain

$$\|S_n f(\cdot, t) - S_n f(\cdot, 0)\|_{L^p(\mathbb{R}^{d-1}; X)} \leq C 2^{\frac{1+\gamma}{p}n} |t| 2^n \|M(\|S_n f\|^r)\|_{L^{p/r}(\mathbb{R}^d, w; X)}^{1/r},$$

where $r \in (0, p)$ and M is the Hardy-Littlewood maximal operator. Choosing r such that $w \in A_{p/r}$, and using that M is bounded on $L^q(\mathbb{R}^d, v)$ for $v \in A_q$ and $q \in (1, \infty)$, we get

$$\|S_n f(\cdot, t) - S_n f(\cdot, 0)\|_{L^p(\mathbb{R}^{d-1}; X)} \leq C 2^{\frac{1+\gamma}{p}n} |t| 2^n \|S_n f\|_{L^p(\mathbb{R}^d, w; X)}, \quad n \in \mathbb{N}_0.$$

Now as in [42] it can be shown that $\lim_{t \rightarrow 0} \tilde{f}(\cdot, t) = \tilde{f}(\cdot, 0)$ in $L^p(\mathbb{R}^{d-1}; X)$, which finishes the proof. \square

Remark 4.3. For $\gamma > 0$ and $w(x', t) = |t|^\gamma$ it follows from Theorem 3.3 that

$$B_{p,1}^{\frac{1+\gamma}{p}}(\mathbb{R}^d, w; X) \hookrightarrow B_{p,1}^{\frac{1}{p}}(\mathbb{R}^d; X).$$

Thus for these exponents the above result can also be deduced from this embedding and the unweighted case $\gamma = 0$. This argument does not work for $\gamma \in (-1, 0)$.

After this preparation we determine the trace space of a Besov space in the general case. We follow the arguments of [42, Theorem 4.9].

Proposition 4.4. *Let X be a Banach space, $d \geq 2$, $p \in (1, \infty)$, $q \in [1, \infty]$, $w(x', t) = |t|^\gamma$ with $\gamma > -1$ and $s > \frac{1+\gamma}{p}$. Then tr as in Proposition 4.2 is a continuous and surjective operator*

$$\text{tr} : B_{p,q}^s(\mathbb{R}^d, w; X) \rightarrow B_{p,q}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)$$

There exists a continuous right-inverse ext of tr which is independent of s, p, q, γ and X .

Remark 4.5. For $p = q \in (1, \infty)$ and $w \in A_p$ the result was shown in [22, Théorème 7.1]. Using an atomic approach, in [24, Theorem 3.5] the trace spaces in the case of radial power weights are determined, also assuming an A_p -condition. For weights $w(x', t) = |t|^\gamma$ the Fourier analytic arguments from [42] that are extended in the proof below do not require this assumption.

Proof of Proposition 4.4. Step 1. We show the continuity of tr . Let $(\varphi_n)_{n \geq 0} \in \Phi(\mathbb{R}^d)$ and $(\phi_n)_{n \geq 0} \in \Phi(\mathbb{R}^{d-1})$. For $f \in \mathcal{S}'(\mathbb{R}^d; X)$ and $g \in \mathcal{S}'(\mathbb{R}^{d-1}; X)$ we write

$$S_n f = \varphi_n * f, \quad T_n g = \phi_n * g, \quad n \geq 0, \quad S_{-1} \equiv 0, \quad T_{-1} \equiv 0.$$

Take $f \in B_{p,q}^s(\mathbb{R}^d, w; X)$, and write $s_0 = s - \frac{1+\gamma}{p} > 0$. In [42] it is shown that

$$\|\text{tr} f\|_{B_{p,q}^{s_0}(\mathbb{R}^{d-1}; X)} = \left\| \sum_{n=0}^{\infty} S_n f(x', 0) \right\|_{B_{p,q}^{s_0}(\mathbb{R}^{d-1}; X)} \leq C \sum_{n=0}^{\infty} \|(2^{ls_0} \|S_{n+l-1} f(x', 0)\|_{L^p(\mathbb{R}^{d-1}; X)})_{l \geq 0}\|_{\ell^q},$$

where the constant C does not depend on f . Note that $\text{supp } \mathcal{F}(S_{n+l-1} f) \subseteq \{|\xi| \leq 2^{n+l}\}$. Applying Lemma 4.1, we get

$$\|\text{tr} f\|_{B_{p,q}^{s_0}(\mathbb{R}^{d-1}; X)} \leq C \sum_{n=0}^{\infty} \left\| (2^{ls_0} 2^{(n+l)\frac{1+\gamma}{p}} \|S_{n+l-1} f\|_{L^p(\mathbb{R}^d, w; X)})_{l \geq 0} \right\|_{\ell^q}$$

$$\begin{aligned}
&\leq C \sum_{n=0}^{\infty} 2^{-ns_0} \left\| (2^{(n+l)s} \|S_{n+l-1}f\|_{L^p(\mathbb{R}^d, w; X)})_{l \geq 0} \right\|_{\ell_q} \\
&\leq C \|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)}.
\end{aligned}$$

Step 2. We define the right-inverse ext as in [49, Section 2.7.2] and [42]. We take $(\rho_n)_{n \geq 0} \in \Phi(\mathbb{R})$ such that $\rho_n(0) = 2^n$ for all n and set for $g \in B_{p,q}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)$

$$\text{ext } g(x', t) := \sum_{n=0}^{\infty} 2^{-n} \rho_n(t) T_n g(x') \quad \text{in } \mathcal{S}'(\mathbb{R}^d; X).$$

This formula does not depend on the parameters. Moreover, $\text{ext } g$ is well-defined in $L^p(\mathbb{R}^d, w; X)$ (and hence in the sense of distributions) since

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^{-n} \|\rho_n T_n g\|_{L^p(\mathbb{R}^d, w; X)} &= \sum_{n=0}^{\infty} 2^{-n} \|\rho_n\|_{L^p(\mathbb{R}, |t|^\gamma dt)} \|T_n g\|_{L^p(\mathbb{R}^{d-1}; X)} \\
&\leq \sum_{n \geq 0} C 2^{-n(\gamma+1)} \|g\|_{L^p(\mathbb{R}^{d-1}; X)} \\
&\leq C \|g\|_{B_{p,q}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)},
\end{aligned}$$

where we used $\gamma > -1$ and $s > \frac{1+\gamma}{p}$. Since $w \in A_\infty$ we can find $r \in (0, \min\{p, q\})$ such that $w \in A_{p/r}$. Let $g \in B_{p,q}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)$. It follows from [36, Proposition 2.4] that

$$\begin{aligned}
\|S_l \text{ext } g\|_{L^p(\mathbb{R}^d, w; X)} &\leq \sum_{j=-1,0,1} \|S_l(2^{-(l+j)} \rho_{l+j}(T_{l+j}g))\|_{L^p(\mathbb{R}^d, w; X)} \\
&\leq C \sum_{j=-1,0,1} 2^{-(l+j)} \|\rho_{l+j}\|_{L^p(\mathbb{R}, |\cdot|^\gamma)} \|T_{l+j}g\|_{L^p(\mathbb{R}^{d-1}; X)}.
\end{aligned}$$

From $\rho_k = 2^{k-1} \rho_1(2^{k-1} \cdot)$ we obtain that $\|\rho_{l+j}\|_{L^p(\mathbb{R}, |\cdot|^\gamma; X)} = 2^{(l+j-1)(1-\frac{1+\gamma}{p})} \|\rho_1\|_{L^p(\mathbb{R}, |\cdot|^\gamma)}$, which leads to

$$\|S_l \text{ext } g\|_{L^p(\mathbb{R}^d, w; X)} \leq C \sum_{j=-1,0,1} 2^{-(l+j-1)\frac{1+\gamma}{p}} \|T_{l+j}g\|_{L^p(\mathbb{R}^{d-1}; X)}.$$

Therefore,

$$\|\text{ext } g\|_{B_{p,q}^s(\mathbb{R}^d, w; X)} \leq C \sum_{j=-1,0,1} \|(2^{l(s-\frac{1+\gamma}{p})q} \|T_{l+j}g\|_{L^p(\mathbb{R}^{d-1}; X)}^q)_{l \geq 0}\|_{\ell_q} \leq C \|g\|_{B_{p,q}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)}.$$

It follows that ext is continuous as asserted, independent of the parameters.

Finally, by the choice of ρ_n we have $\text{tr} \circ \text{ext } g = g$ for all $g \in \mathcal{S}(\mathbb{R}^d; X)$. By [36, Lemma 3.8], this space is dense in $B_{p,q}^s(\mathbb{R}^d, w; X)$ if $q < \infty$, and the identity extends to all $g \in B_{p,q}^s(\mathbb{R}^d, w; X)$. For $q = \infty$ we have $B_{p,\infty}^s(\mathbb{R}^d, w; X) \hookrightarrow B_{p,1}^{s-\varepsilon}(\mathbb{R}^d, w; X)$ for all $\varepsilon > 0$, and thus ext is also a right-inverse in this case. \square

Remark 4.6. Analogous to Remark 4.3, in case $\gamma > 0$ the continuity of the trace can also be obtained by a simple argument from the unweighted case and the embeddings from Theorem 3.3. Indeed, for $s > \frac{1+\gamma}{p}$ one has

$$\|\text{tr } f\|_{B_{p,q}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)} \leq C \|f\|_{B_{p,q}^{s-\frac{\gamma}{p}}(\mathbb{R}^d; X)} \leq C \|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)},$$

since $s - \frac{d+\gamma}{p} = s - \frac{\gamma}{p} - \frac{d}{p}$ and $0 \leq \frac{\gamma}{p}$. Similarly, with these arguments one can give a simple argument for the continuity of the extension operator for $\gamma \in (-1, 0)$.

4.3. The trace space for Triebel-Lizorkin, Bessel potential and Sobolev spaces. We now determine the trace space of vector-valued F -, H -, and W -spaces with an anisotropic power weight. As known in the unweighted case, for Triebel-Lizorkin spaces the trace maps independently of the microscopic parameter $q \in [1, \infty]$.

In contrast to [42, Lemma 4.15] and [24, Theorem 3.6], our proof is purely Fourier analytic and does not use the atomic approach. The rather short argument is based on the above result for Besov spaces and the Sobolev embeddings for Triebel-Lizorkin spaces from Theorem 3.1. Again an A_p -condition for the weight is not required and it suffices to have $w \in A_\infty$. In the unweighted situation a proof based on atomic decompositions was given in [42]. In the scalar and unweighted case other methods can be used [49, Theorem 2.7.2].

Theorem 4.7. *Let X be a Banach space, $p \in (1, \infty)$, $q \in [1, \infty]$, $w(x', t) = |t|^\gamma$ with $\gamma > -1$ and $s > \frac{1+\gamma}{p}$. Then the trace tr is a continuous and surjective operator*

$$\text{tr} : F_{p,q}^s(\mathbb{R}^d, w; X) \rightarrow B_{p,p}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X).$$

Moreover, there exists a continuous right-inverse ext which is independent of s, p, q, γ and X .

Proof. Step 1. For the continuity of tr it suffices to consider $q = \infty$. Let $f \in F_{p,\infty}^s(\mathbb{R}^d, w; X)$, and set $\tilde{s} = s - \frac{\varepsilon}{p}$, $\tilde{\gamma} = \gamma - \varepsilon$ and $\tilde{w}(x', t) = |t|^{\tilde{\gamma}}$, where $\varepsilon > 0$ is sufficiently small. Since $s - \frac{d+\gamma}{p} = \tilde{s} - \frac{d+\tilde{\gamma}}{p}$ and $\frac{\tilde{\gamma}}{p} < \frac{\gamma}{p}$, we obtain from Theorem 3.3 that

$$\|f\|_{B_{p,p}^{\tilde{s}}(\mathbb{R}^d, \tilde{w}; X)} = \|f\|_{F_{p,p}^{\tilde{s}}(\mathbb{R}^d, \tilde{w}; X)} \leq C \|f\|_{F_{p,\infty}^s(\mathbb{R}^d, w; X)}.$$

Now by Proposition 4.4,

$$\|\text{tr} f\|_{B_{p,p}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)} = \|\text{tr} f\|_{B_{p,p}^{\tilde{s}-\frac{1+\tilde{\gamma}}{p}}(\mathbb{R}^{d-1}; X)} \leq C \|f\|_{B_{p,p}^{\tilde{s}}(\mathbb{R}^d, \tilde{w}; X)}$$

and the result follows if we combine both estimates.

Step 2. For the continuity of the right-inverse it suffices to consider $q = 1$. Take ext as in Proposition 4.4, let $g \in B_{p,p}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)$, and set $\tilde{s} = s + \frac{\varepsilon}{p}$, $\tilde{\gamma} = \gamma + \varepsilon$ and $\tilde{w}(x', t) = |t|^{\tilde{\gamma}}$. Then we infer from Proposition 4.4

$$\|\text{ext } g\|_{B_{p,p}^{\tilde{s}}(\mathbb{R}^d, \tilde{w}; X)} \leq C \|g\|_{B_{p,p}^{\tilde{s}-\frac{1+\tilde{\gamma}}{p}}(\mathbb{R}^{d-1}; X)} = C \|g\|_{B_{p,p}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)}.$$

Since $\tilde{s} - \frac{d+\tilde{\gamma}}{p} = s - \frac{d+\gamma}{p}$ and $\frac{\gamma}{p} < \frac{\tilde{\gamma}}{p}$, Theorem 3.3 implies

$$\|\text{ext } g\|_{F_{p,1}^s(\mathbb{R}^d, w; X)} \leq C \|\text{ext } g\|_{F_{p,p}^{\tilde{s}}(\mathbb{R}^d, \tilde{w}; X)} = C \|\text{ext } g\|_{B_{p,p}^{\tilde{s}}(\mathbb{R}^d, \tilde{w}; X)}$$

and the continuity follows again from the combination of these estimates. The fact that $\text{tr} \circ \text{ext } g = g$ for all $g \in B_{p,p}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)$ is clear from Proposition 4.4. \square

As a consequence of the above result we can determine the trace spaces of the H - and W -spaces under the assumption that $w \in A_p$.

Corollary 4.8. *Let X be a Banach space, $p \in (1, \infty)$, $w(x', t) = |t|^\gamma$ with $\gamma \in (-1, p-1)$ and $s > \frac{1+\gamma}{p}$. Then the trace tr is a continuous and surjective operator*

$$\text{tr} : H^{s,p}(\mathbb{R}^d, w; X) \rightarrow B_{p,p}^{s-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X).$$

If $m \in \mathbb{N}$, then tr is a continuous and surjective operator

$$\text{tr} : W^{m,p}(\mathbb{R}^d, w; X) \rightarrow B_{p,p}^{m-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X).$$

In both cases there is a continuous right-inverse ext which is independent of s, m, p, γ and X .

Proof. This is a consequence of Theorem 4.7 and the embeddings (2.4) and (2.5). \square

Proof of Theorem 1.1. This follows from Proposition 4.4, Theorem 4.7 and Corollary 4.8. \square

5. FOURIER MULTIPLIERS

In this section we derive an operator-valued Fourier multiplier theorem for weighted Besov and Triebel-Lizorkin spaces.

For a compact set $K \subseteq \mathbb{R}^d$, let $L_K^p(\mathbb{R}^d, w; X) = \{f \in L^p(\mathbb{R}^d, w; X) : \text{supp } \hat{f} \subseteq K\}$.

Lemma 5.1. *Let X, Y be Banach spaces, let $p \in (1, \infty)$, $q \in [1, \infty]$ and $w \in A_\infty$. Let $r \in (0, \min\{p, q\})$ be such that $w \in A_{p/r}$. Let $K_0, K_1, \dots \subseteq \mathbb{R}^d$ be compact sets with $\theta_n = \text{diam } K_n > 0$ for all n . Then there is a constant C such that for all $(M_n)_{n \geq 0} \subseteq \mathcal{FL}^1(\mathbb{R}^d; \mathcal{L}(X, Y))$ and all $(f_n)_{n \geq 0} \subseteq L^p(\mathbb{R}^d, w; \ell^q(X))$ with $f_n \in L_{K_n}^p(\mathbb{R}^d, w; X)$ for $n \in \mathbb{N}$ it holds*

$$\begin{aligned} & \|(\mathcal{F}^{-1}(M_n \mathcal{F} f_n))_{n \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(Y))} \\ & \leq C \sup_{k \geq 0} \|(1 + |\cdot|^{d/r}) \mathcal{F}^{-1}(M_k(\theta_k \cdot))\|_{L^1(\mathbb{R}^d; \mathcal{L}(X, Y))} \|(f_n)_{n \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))}. \end{aligned}$$

Proof. In the proof of [10, Proposition 2.2] and [49, Theorem 1.6.3] it is shown that for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$ one has

$$(5.1) \quad \|\mathcal{F}^{-1}(M_n \mathcal{F} f_n)(x)\|_Y \leq C f_n^*(x) \|(1 + |\cdot|^{d/r}) \mathcal{F}^{-1}(M_n(\theta_n \cdot))\|_{L^1(\mathbb{R}^d; \mathcal{L}(X, Y))},$$

where $f_n^*(x) = \sup_{z \in \mathbb{R}^d} \frac{\|f_n(x-z)\|_X}{1 + |\theta_n z|^{d/r}}$ is of Peetre type. Moreover, it is shown in the proof of [10, Lemma 2.1] and [49, Theorem 1.6.2] that $f_n^*(x) \leq C(M\|f_n\|_X^r(x))^{1/r}$ for all x and $r > 0$, where M is as before the Hardy-Littlewood maximal operator. Now from [36, Proposition 2.2] we obtain that

$$\|(M\|f_n\|_X^r)_{n \geq 0}\|_{L^{p/r}(\mathbb{R}^d, w; \ell^q)} \leq C \|(f_n)_{n \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))}.$$

The result follows from taking $L^p(\mathbb{R}^d, w; \ell^q)$ -norms in (5.1). \square

For $m \in L^\infty(\mathbb{R}^d; \mathcal{L}(X, Y))$, let the operator $T_m : \mathcal{S}(\mathbb{R}^d; X) \rightarrow \mathcal{S}'(\mathbb{R}^d; X)$ be given by $T_m f = \mathcal{F}^{-1}(m \mathcal{F} f)$. The following result gives a sufficient condition on m for the extension of T_m to weighted Besov and Triebel-Lizorkin spaces.

Proposition 5.2. *Let X, Y be Banach spaces, $p \in (1, \infty)$, $q \in [1, \infty]$, $w \in A_\infty$ and $s \in \mathbb{R}$. Let $r \in (0, \min\{p, q\})$ be such that $w \in A_{p/r}$. Let $\mathcal{A} \in \{F, B\}$. Assume $m \in C^\infty(\mathbb{R}^d; \mathcal{L}(X, Y))$ is such that $D^\alpha m$ grows at most polynomially at infinity for all $\alpha \in \mathbb{N}_0^d$ and satisfies*

$$(5.2) \quad \sup_{|\alpha| \leq d + \lceil d/r \rceil + 1} \sup_{\xi \in \mathbb{R}^d} \|(1 + |\xi|)^{|\alpha|} D^\alpha m(\xi)\|_{\mathcal{L}(X, Y)} = K_m < \infty.$$

Then $T_m = \mathcal{F}^{-1} m \mathcal{F}$ extends to a continuous operator from $\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)$ to $\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; Y)$. The operator norm of T_m is bounded by CK_m , where C does not depend on m .

Remark 5.3.

- (i) In the scalar case Fourier multiplier results of the above type can be found in [49].
- (ii) An extensive treatment of operator-valued Fourier multipliers on Besov spaces can be found in [4, 21, 26].
- (iii) The condition on m guarantees that $m \hat{f}$ and hence $T_m f$ is a well-defined element of $\mathcal{S}'(\mathbb{R}^d; X)$ for all $f \in \mathcal{S}'(\mathbb{R}^d; X)$, see [43, Section 3]. In the sequel we only consider symbols which satisfy these requirements. One can see that it suffices to have that $m \in C^{d + \lceil d/r \rceil + 1}(\mathbb{R}^d; \mathcal{L}(X, Y))$ and that the above Mihlin condition holds true.
- (iv) Results on operator-valued Fourier multipliers in Triebel-Lizorkin space can be found in [10]. Note that in the unweighted situation in [10, Proposition 3.6] the important case $q = \infty$ was excluded, due to the fact that $\mathcal{S}(\mathbb{R}^d; X)$ is not dense in the corresponding function space.

Proof of Proposition 5.2. Let us consider the case of F -spaces. Let $(\varphi_n)_{n \geq 0} \in \Phi(\mathbb{R}^d)$ and $\varphi_{-1} = 0$. Since $f = \sum_{n=0}^\infty S_n f$, for $f \in F_{p,q}^s(\mathbb{R}^d, w; X)$ we have

$$\|\mathcal{F}^{-1} m \mathcal{F} f\|_{F_{p,q}^s(\mathbb{R}^d, w; Y)} \leq \sum_{l=-1,0,1} \|([\mathcal{F}^{-1} \hat{\varphi}_n m \mathcal{F}] 2^{sn} \mathcal{F}^{-1} \hat{\varphi}_{n+l} \mathcal{F} f)_{n \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(Y))}.$$

For fixed l , let $\mathcal{F}^{-1}\widehat{\varphi}_{n+l}\mathcal{F}$. Then $\text{supp } \widehat{f}_n \subset \{|\xi| \leq 2^{n+1}\}$. Since $\widehat{\varphi}_k(2^{k+1}\cdot) = \widehat{\varphi}_1(4\cdot)$, in order to apply Lemma 5.1 we have to show that

$$(5.3) \quad \sup_{k \geq 0} \|(1 + |\cdot|^{d/r})\mathcal{F}^{-1}(\widehat{\varphi}_1(4\cdot)m(2^{k+1}\cdot))\|_{L^1(\mathbb{R}^d; \mathcal{L}(X,Y))} < \infty.$$

Let j be the smallest even number which satisfies $j > d + \frac{d}{r}$. Using that $\mathcal{F}^{-1} : L^1 \rightarrow L^\infty$ is bounded, we get

$$(5.4) \quad \begin{aligned} \|(1 + |\cdot|^j)\mathcal{F}^{-1}(\widehat{\varphi}_1(4\cdot)m(2^{k+1}\cdot))\|_{L^\infty(\mathbb{R}^d; \mathcal{L}(X,Y))} \\ \leq \|(1 + (-\Delta)^{j/2})[\widehat{\varphi}_1(4\cdot)m(2^{k+1}\cdot)]\|_{L^1(\mathbb{R}^d; \mathcal{L}(X,Y))}. \end{aligned}$$

By the Leibniz rule, in the above norm the derivatives consists of a finite linear combination of terms of the form

$$g_{\alpha,\beta} := D^\alpha \widehat{\varphi}_1(4\cdot) 2^{|\beta|(k+1)} (D^\beta m)(2^{k+1}\cdot),$$

where α, β are multiindices with $|\alpha| + |\beta| \leq j$. Since $\widehat{\varphi}_1(4\cdot)$ is supported in the unit ball, it follows from (5.2) that each $g_{\alpha,\beta}$ is bounded independently of k . Hence the right-hand side of (5.4) is bounded by cK_m . Hence

$$\|(1 + |\cdot|^{d/r})\mathcal{F}^{-1}(\widehat{\varphi}_1(4\cdot)m(2^{k+1}\cdot))\|_{L^1(\mathbb{R}^d; \mathcal{L}(X,Y))} \leq cK_m \|(1 + |\cdot|^{-j+\frac{d}{r}})\|_{L^1(\mathbb{R}^d)} = CK_m,$$

and (5.3) follows.

The case of a B -space can be treated in the same way, using Lemma 5.1 on each $\{|\xi| \leq 2^{k+1}\}$ separately. \square

We record an important consequence of the multiplier result. For a definition and properties of the H^∞ -calculus of sectorial operators we refer to [30]. Sectoriality is defined in Section 2.2.

Corollary 5.4. *Let X be a Banach space, $p \in (1, \infty)$, $q \in [1, \infty]$, $w \in A_\infty$ and $s \in \mathbb{R}$. Let $\mathcal{A} \in \{F, B\}$. Then the following assertions hold.*

- (1) *The realization of ∂_t with domain $\mathcal{A}_{p,q}^{s+1}(\mathbb{R}, w; X)$ on $\mathcal{A}_{p,q}^s(\mathbb{R}, w; X)$ is sectorial with spectral angle equal to $\frac{\pi}{2}$ and has a bounded \mathcal{H}^∞ -calculus on each sector Σ_θ with $\theta > \frac{\pi}{2}$.*
- (2) *The realization of $-\Delta$ with domain $\mathcal{A}_{p,q}^{s+2}(\mathbb{R}, w; X)$ on $\mathcal{A}_{p,q}^s(\mathbb{R}, w; X)$ is sectorial with spectral angle equal to zero and has a bounded \mathcal{H}^∞ -calculus on each sector Σ_θ with $\theta > 0$.*

Proof. As in [30, Example 10.2] one can define the holomorphic functional calculus via Fourier transform and show its boundedness by using Proposition 5.2. Note that any symbol m arising in this context is smooth and satisfies $\sup_{\xi \in \mathbb{R}^d} |(1 + |\xi|)^{|\alpha|} D^\alpha m(\xi)| < \infty$ for any multiindex α .

For $-\Delta$ the explicit description of the domain is a direct consequence of the lifting property of the weighted F - and B -spaces (see [36, Proposition 3.9]). For ∂_t one uses the lifting property and Proposition 5.2. \square

6. CHARACTERIZATION BY DIFFERENCES

For a weight w and an integer $m \geq 1$ define

$$\Delta_h^m f(x) = \sum_{\ell=0}^m \binom{m}{\ell} (-1)^\ell f(x + (m - \ell)h), \quad x, h \in \mathbb{R}^d.$$

For $f \in L^p(\mathbb{R}^d, w; X)$, let further

$$[f]_{F_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} = \left\| \left(\int_0^\infty t^{-sq} \left(t^{-d} \int_{|h| \leq t} \|\Delta_h^m f\|_X dh \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d, w)},$$

with obvious modifications if $q = \infty$, and

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} := \|f\|_{L^p(\mathbb{R}^d, w; X)} + [f]_{F_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)}.$$

We also write $\|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)}$ for $\|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)}$ if there is no danger of confusion. Note that if $q = 1$, then Fubini's theorem yields

$$[f]_{F_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} = c_d \left\| \int_{\mathbb{R}^d} |h|^{-s-d} \|\Delta_h^m f\|_X dh \right\|_{L^p(\mathbb{R}^d, w)}$$

One can extend a well-known result on the equivalence of norms to the weighted case (cf. [44, Proposition 6], [49, Section 2.5.10] and [52, Theorem 6.9]).

Proposition 6.1. *Let X be a Banach space, $s > 0$, $p \in (1, \infty)$, $q \in [1, \infty]$ and $w \in A_p$. Let $m \geq 1$ be an integer such that $m > s$. Then there is a constant $C > 0$ such that for all $f \in L^p(\mathbb{R}^d, w; X)$*

$$(6.1) \quad C^{-1} \|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)} \leq \|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} \leq C \|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)},$$

whenever one of these expressions is finite.

To state a similar result for Besov spaces, for $f \in L^p(\mathbb{R}^d, w; X)$ let

$$[f]_{B_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} = \left(\int_0^\infty t^{-sq} \left\| t^{-d} \int_{|h| \leq t} \|\Delta_h^m f\|_X dh \right\|_{L^p(\mathbb{R}^d, w)}^q \frac{dt}{t} \right)^{1/q},$$

again with obvious modifications if $q = \infty$, and

$$\|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} := \|f\|_{L^p(\mathbb{R}^d, w; X)} + [f]_{B_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)}.$$

Then the following holds true.

Proposition 6.2. *Let X be a Banach space, $s > 0$, $p \in (1, \infty)$, $q \in [1, \infty]$ and $w \in A_p$. Let $m \in \mathbb{N}$ be such that $m > s$. There is a constant $C > 0$ such that for all $f \in L^p(\mathbb{R}^d, w; X)$*

$$(6.2) \quad C^{-1} \|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)} \leq \|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} \leq C \|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)},$$

whenever one of these expressions is finite.

Remark 6.3. Define the $L^p(\mathbb{R}^d, w; X)$ -modulus of smoothness as

$$\omega_{p,w}^m(f, t) := \sup_{|h| \leq t} \|\Delta_h^m f\|_{L^p(\mathbb{R}^d, w; X)}, \quad t > 0.$$

In the unweighted case $w \equiv 1$, the norm

$$\|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} := \|f\|_{L^p(\mathbb{R}^d, w; X)} + \left(\int_0^\infty t^{-sq} \omega_{p,w}^m(f, t)^q \frac{dt}{t} \right)^{1/q},$$

defines an equivalent norm on $B_{p,q}^s(\mathbb{R}^d, w; X)$ if $m > s$ (modification if $q = \infty$). We do not know if this extends to the weighted setting. However, by Minkowski's inequality one has

$$\left\| t^{-d} \int_{|h| \leq t} \|\Delta_h^m f\|_X dh \right\|_{L^p(\mathbb{R}^d, w)} \leq t^{-d} \int_{|h| \leq t} \|\Delta_h^m f\|_{L^p(\mathbb{R}^d, w; X)} dh \leq C \sup_{|h| \leq t} \|\Delta_h^m f\|_{L^p(\mathbb{R}^d, w; X)}.$$

Therefore, one always has that $\|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)}^{(m)} \leq C \|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)}$.

7. MIXED DERIVATIVE EMBEDDINGS

7.1. The general case. For fractional powers of sectorial operators we refer to Section 2.2.

Theorem 7.1. *Let X be a Banach space, $p \in (1, \infty)$, $q \in [1, \infty]$, $w \in A_\infty$, $s \in \mathbb{R}$ and $\alpha > 0$. Let A be a sectorial operator on X with spectral angle $\phi_A < \pi$, and let $\mathcal{A} \in \{F, B\}$. Then for all $\theta \in [0, 1]$ one has*

$$\mathcal{A}_{p,q}^{s+\alpha}(\mathbb{R}^d, w; X) \cap \mathcal{A}_{p,q}^s(\mathbb{R}^d, w; D(A)) \hookrightarrow \mathcal{A}_{p,q}^{s+\theta\alpha}(\mathbb{R}^d, w; D(A^{1-\theta})).$$

Remark 7.2.

- (i) In the literature embeddings as in the above theorem are often proved using the so-called mixed derivative theorem due to [48]. We give a direct proof using Proposition 5.2. Embeddings of this type are widely used in the context of parabolic evolution equations, in particular when inhomogeneous boundary conditions are considered, see e.g. [15, 16, 18, 28, 35].
- (ii) For $d = 1$, corresponding results for intersection spaces on the half-line with H - and W -regularity are proved in [16] in the unweighted case, and in [34] for power weights $w(t) = |t|^\gamma$ with $\gamma \in [0, p-1)$.

Proof of Theorem 7.1. Without loss of generality we may assume that A is invertible.

For all $\alpha > 0$ the realization of $(1 - \Delta)^{\alpha/2}$ on $\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)$ is sectorial with spectral angle equal to zero by Corollary 5.4. As in the proof of Corollary 5.4, it is a consequence of the lifting property of the underlying spaces that the domain of this operator equals $\mathcal{A}_{p,q}^{s+\alpha}(\mathbb{R}^d, w; X)$. It is further straightforward to see that the (pointwise) realization of A with domain $\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; D(A))$ on $\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)$ is sectorial with spectral angle at most ϕ_A .

Step 1. We show that $(1 - \Delta)^{\alpha/2} + A$ is an isomorphic mapping

$$\mathcal{A}_{p,q}^{s+\alpha}(\mathbb{R}^d, w; X) \cap \mathcal{A}_{p,q}^s(\mathbb{R}^d, w; D(A)) \rightarrow \mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X).$$

The boundedness of $(1 - \Delta)^{\alpha/2} + A$ is clear. For the boundedness of its inverse, let $f \in \mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)$ and consider the equation

$$(7.1) \quad ((1 - \Delta)^{\alpha/2} + A)u = f \quad \text{in } \mathcal{S}'(\mathbb{R}^d; X)$$

One can check that for all $\beta \in \mathbb{N}_0^d$ the symbol $m(\xi) = A((1 + |\xi|^2)^{\alpha/2} + A)^{-1}$ satisfies

$$\sup_{\xi \in \mathbb{R}^d} \|(1 + |\xi|)^{|\beta|} D^\beta m(\xi)\|_{\mathcal{L}(X)} \leq C_\beta < \infty.$$

Hence the unique solution $u \in \mathcal{S}'(\mathbb{R}^d; D(A))$ of (7.1) is given by $u = \mathcal{F}^{-1}((1 + |\cdot|^2)^{\alpha/2} + A)^{-1} \mathcal{F}f$. From Proposition 5.2 we obtain that

$$\|Au\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)} \leq C\|f\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)}.$$

Since A is invertible, we conclude $\|u\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; D(A))} \leq C\|f\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)}$. Moreover, using this estimate and (7.1), it follows that

$$\|(1 - \Delta)^{\alpha/2} u\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)} \leq C\|f\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)}.$$

Hence also $\|u\|_{\mathcal{A}_{p,q}^{s+\alpha}(\mathbb{R}^d, w; X)} \leq C\|f\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)}$. This completes the proof of the claim.

Step 2. By the invertibility of the operator,

$$u \mapsto \|((1 - \Delta)^{\alpha/2} + A)u\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)}$$

defines an equivalent norm on $\mathcal{A}_{p,q}^{s+\alpha}(\mathbb{R}^d, w; X) \cap \mathcal{A}_{p,q}^s(\mathbb{R}^d, w; D(A))$. Further, for all $\theta \in (0, 1)$ we have that

$$u \mapsto \|(1 - \Delta)^{\theta\alpha/2} A^{1-\theta} u\|_{\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)}$$

defines an equivalent norm on $\mathcal{A}_{p,q}^{s+\theta\alpha}(\mathbb{R}^d, w; D(A^{1-\theta}))$. To show the asserted embedding it thus suffices to prove that

$$(1 - \Delta)^{\theta\alpha/2} A^{1-\theta} ((1 - \Delta)^{\alpha/2} + A)^{-1}$$

is bounded on $\mathcal{A}_{p,q}^s(\mathbb{R}^d, w; X)$. As above we can rewrite this operator into the form $\mathcal{F}^{-1}m\mathcal{F}$, where now the symbol m is given by

$$m(\xi) = (1 + |\xi|^2)^{\theta\alpha/2} A^{1-\theta} ((1 + |\xi|^2)^{\alpha/2} + A)^{-1}.$$

By [32, Proposition 2.2.15], for all $\lambda \geq 1$ and $x \in X$ we obtain

$$\|A^{1-\theta}(\lambda + A)^{-1}x\|_X \leq C\|(\lambda + A)^{-1}x\|_{D(A)}^{1-\theta}\|(\lambda + A)^{-1}x\|_X^\theta \leq C\lambda^{-\theta}\|x\|_X.$$

Using this, one can verify the conditions of Proposition 5.2 for the symbol m . \square

7.2. Refined embeddings. In the case where X and $D(A)$ are F -spaces we can improve the above embeddings within the inner regularity scale. The next result includes in particular Theorem 1.3.

Theorem 7.3. *Let X be a Banach space, $d, m \in \mathbb{N}$, $p, p_0, p_1, p_2 \in (1, \infty)$, $q \in [1, \infty]$, $s_1, s_2 \in \mathbb{R}$, $\alpha_1, \alpha_2 > 0$ and $\theta \in (0, 1)$ with*

$$\frac{1}{p_2} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

Let $w \in A_\infty$ on \mathbb{R}^d and $w_0, w_1, w_2 \in A_\infty$ on \mathbb{R}^m with $w_2 = w_0^{1-\theta} w_1^\theta$, and let $\mathcal{A} \in \{F, B\}$. Let the spaces \mathbb{X}, \mathbb{Y} and \mathbb{F} be given by

$$\mathbb{X} = \mathcal{A}_{p,q}^{s_1+\alpha_1}(\mathbb{R}^d, w; F_{p_0,\infty}^{s_2}(\mathbb{R}^m, w_0; X)), \quad \mathbb{Y} = \mathcal{A}_{p,q}^{s_1}(\mathbb{R}^d, w; F_{p_1,\infty}^{s_2+\alpha_2}(\mathbb{R}^m, w_1; X)).$$

$$\mathbb{F} = \mathcal{A}_{p,q}^{r_1}(\mathbb{R}^d, w; F_{p_2,1}^{r_2}(\mathbb{R}^m, w_2; X)), \quad \text{with } r_1 = s_1 + \theta\alpha_1 \text{ and } r_2 = s_2 + (1-\theta)\alpha_2.$$

Then $\mathbb{X} \cap \mathbb{Y} \hookrightarrow \mathbb{F}$, and there is a constant C such that for all $f \in \mathbb{X} \cap \mathbb{Y}$

$$\|f\|_{\mathbb{F}} \leq C \|f\|_{\mathbb{X}}^\theta \|f\|_{\mathbb{Y}}^{1-\theta}.$$

In particular, note that one can take $p_0 = p_1 = p_2$ and $w_0 = w_1 = w_2$ in the above result.

Proof. We write $x_1 \in \mathbb{R}^d$, $x_2 \in \mathbb{R}^m$ and $f(x_1, x_2)$. We have

$$\|f\|_{\mathbb{F}} = \left\| x_1 \mapsto \left\| \left(2^{n(s_1+\theta\alpha_1)} \|S_n f(x_1, \cdot)\|_{F_{p_2,1}^{s_2+(1-\theta)\alpha_2}(\mathbb{R}^m, w_2; X)} \right)_{n \geq 0} \right\|_{\ell^q} \right\|_{L^p(\mathbb{R}^d, w)}.$$

The Gagliardo-Nirenberg type inequality for F -spaces from [36, Proposition 5.1] implies that

$$\|S_n f(x_1, \cdot)\|_{F_{p_2,1}^{s_2+(1-\theta)\alpha_2}(\mathbb{R}^m, w_2; X)} \leq C \|S_n f(x_1, \cdot)\|_{F_{p_0,\infty}^{s_2}(\mathbb{R}^m, w_0; X)}^\theta \|S_n f(x_1, \cdot)\|_{F_{p_1,\infty}^{s_2+\alpha_2}(\mathbb{R}^m, w_1; X)}^{1-\theta}.$$

Therefore, using Hölder's inequality in ℓ^q and in $L^p(\mathbb{R}^d, w)$, we get

$$\begin{aligned} \|f\|_{\mathbb{F}} &\leq C \left\| \left[\left(2^{n(s_1+\alpha_1)} \|S_n f\|_{F_{p_0,\infty}^{s_2}(\mathbb{R}^m, w_0; X)} \right)^\theta \left(2^{ns_1} \|S_n f\|_{F_{p_1,\infty}^{s_2+\alpha_2}(\mathbb{R}^m, w_1; X)} \right)^{1-\theta} \right]_{n \geq 0} \right\|_{L^p(\mathbb{R}^d, w; \ell^q)} \\ &\leq C \left\| \left(2^{n(s_1+\alpha_1)} \|S_n f\|_{F_{p_0,\infty}^{s_2}(\mathbb{R}^m, w_0; X)} \right)_{n \geq 0} \right\|_{L^p(\mathbb{R}^d, w; \ell^q)}^\theta \\ &\quad \times \left\| \left(2^{ns_1} \|S_n f\|_{F_{p_1,\infty}^{s_2+\alpha_2}(\mathbb{R}^m, w_1; X)} \right)_{n \geq 0} \right\|_{L^p(\mathbb{R}^d, w; \ell^q)}^{1-\theta} = C \|f\|_{\mathbb{X}}^\theta \|f\|_{\mathbb{Y}}^{1-\theta}. \end{aligned}$$

This shows the asserted inequality. The embedding follows from Young's inequality. The case of $\mathcal{A} = B$ is treated in the same way. \square

Remark 7.4.

- (i) The above result improves Theorem 7.1 not only with respect to the microscopic parameter in the inner scale. It also gives a multiplicative estimate with powers corresponding to θ instead of an additive estimate, to which Young's inequality "with ε " can be applied.
- (ii) For $w \in A_p$ one can use the elementary embeddings (2.4) and (2.5) between H -, W - and F -spaces to obtain a variety of embeddings as above with possibly different regularities for the inner spaces. For instance, it follows that for all $p, q \in (1, \infty)$ and $s \in \mathbb{R}$

$$B_{p,p}^{s+\alpha}(\mathbb{R}; L^q(\mathbb{R}^d; X)) \cap B_{p,p}^s(\mathbb{R}; B_{q,q}^\beta(\mathbb{R}^d; X)) \hookrightarrow B_{p,p}^{s+\theta\alpha}(\mathbb{R}; H^{(1-\theta)\beta, q}(\mathbb{R}^d; X))$$

for an arbitrary Banach space X .

- (iii) Combining the theorem with real interpolation techniques as in [16, 28, 34] yields mixed derivative embeddings with differing spaces also for the outer regularities. Taking into account embeddings based on type and cotype of the underlying spaces as in [54, Proposition 3.1] gives even more flexibility.

7.3. A counterexample. It is natural to ask for improvements of Theorem 7.3 with respect to the microscopic parameter in the outer scale. The next result implies that this is not possible.

Proposition 7.5. *Let $s_1, s_2 \in \mathbb{R}$, $\alpha_1, \alpha_2 > 0$, $p, p_0 \in (1, \infty)$, $1 \leq r < q \leq \infty$ and $\theta \in [0, 1]$. Then*

$$F_{p,q}^{s_1+\alpha_1}(\mathbb{R}; B_{p_0,1}^{s_2}(\mathbb{R})) \cap F_{p,q}^{s_1}(\mathbb{R}; B_{p_0,1}^{s_2+\alpha_2}(\mathbb{R})) \not\subseteq F_{p,r}^{s_1+\theta\alpha_1}(\mathbb{R}; B_{p_0,\infty}^{s_2+\alpha_2(1-\theta)}(\mathbb{R})).$$

Remark 7.6. Note that, as a consequence of Proposition 7.5 and the monotonic properties (2.6) of the function spaces, the inclusion of intersection spaces where the Besov space in the inner scale is replaced by any F -, B - or H -space with the same smoothness and integrability exponent does not hold as well.

Proof. In order to obtain a contradiction, assume that the inclusion holds, which is then a continuous embedding by a closed graph argument.

Let $R = 2^{\alpha_1/\alpha_2}$. Recall from [49, Remark 2.3.1/3] that for $t \in \mathbb{R}$ and $z \in [1, \infty]$, the norm of $B_{p_0,z}^t(\mathbb{R})$ is equivalent to

$$f \mapsto \|(R^{tn}\psi_n * f)_{n \geq 0}\|_{\ell^z(L^{p_0})},$$

where $(\psi_n)_{n \geq 0}$ is a decomposition of the identity with the 2^n -factor replaced by R^n (see the definition of $\Phi(\mathbb{R})$). Let further $(\varphi_n)_{n \geq 0} \in \Phi(\mathbb{R})$.

We may assume that there is $\delta > 0$ such that $\widehat{\varphi}_n = 1$ and $\widehat{\varphi}_j = 0$ for $j \neq n$ on $[2^n - \delta, 2^n + \delta]$, and $\widehat{\psi}_n = 1$ and $\widehat{\psi}_j = 0$ for $j \neq n$ on $[R^n - \delta, R^n + \delta]$, for all $n \geq 1$. Fix a sequence of real numbers $(a_n)_{n \geq 1}$ of which only finitely many are nonzero. Let $u : \mathbb{R} \rightarrow B_{p_0,1}^{t+\alpha_2}(\mathbb{R})$ be defined by

$$u(x)(y) = \sum_{n \geq 1} a_n (\mathcal{F}^{-1} \mathbf{1}_{[2^n - \delta, 2^n + \delta]})(x) (\mathcal{F}^{-1} \mathbf{1}_{[R^n - \delta, R^n + \delta]})(y) = \sum_{n \geq 1} a_n e^{i2^n x} e^{iR^n y} \zeta(x) \zeta(y).$$

Let $Y = B_{p_0,z}^t(\mathbb{R})$ and $s \in \mathbb{R}$. Then one has

$$\|u\|_{F_{p,q}^s(\mathbb{R}; Y)} = \|(2^{js} u * \varphi_j)_{j \geq 0}\|_{L^p(\mathbb{R}; \ell^q(Y))}.$$

Moreover, for each $j \geq 0$ and $x \in \mathbb{R}$,

$$\|u * \varphi_j(x)\|_Y = \|(R^{nt} \psi_n * (u * \varphi_j(x)))_{n \geq 0}\|_{\ell^z(L^{p_0}(\mathbb{R}))} = R^{jt} |a_j| \|\zeta(x)\|_{L^{p_0}(\mathbb{R})}.$$

It follows that

$$\|u\|_{F_{p,q}^s(\mathbb{R}; B_{p_0,z}^t(\mathbb{R}))} = \|u\|_{F_{p,q}^s(\mathbb{R}; Y)} \approx C_{p,p_0} \|(2^{js} R^{jt} a_j)_{j \geq 1}\|_{\ell^q},$$

where $C_{p,p_0} = \|\zeta\|_{L^p(\mathbb{R})} \|\zeta\|_{L^{p_0}(\mathbb{R})}$ and $A \approx B$ means there is a constant $c > 0$ independent of $(a_j)_{j \geq 1}$ such that $c^{-1}B \leq A \leq cB$. Therefore,

$$\|u\|_{F_{p,r}^{s_1+\theta\alpha_1}(\mathbb{R}; B_{p_0,\infty}^{s_2+(1-\theta)\alpha_2}(\mathbb{R}))} \approx \|(2^{j(s_1+\theta\alpha_1)} R^{j(s_2+(1-\theta)\alpha_2)} a_j)_{j \geq 1}\|_{\ell^r} = \|(2^{js_1} 2^{\frac{\alpha_1}{\alpha_2} j s_2} 2^{j\alpha_1} a_j)_{j \geq 1}\|_{\ell^r}$$

$$\|u\|_{F_{p,q}^{s_1+\alpha_1}(\mathbb{R}; B_{p_0,1}^{s_2}(\mathbb{R}))} \approx \|(2^{j(s_1+\alpha_1)} R^{js_2} a_j)_{j \geq 1}\|_{\ell^q} = \|(2^{js_1} 2^{\frac{\alpha_1}{\alpha_2} j s_2} 2^{j\alpha_1} a_j)_{j \geq 1}\|_{\ell^q},$$

$$\|u\|_{F_{p,q}^{s_1}(\mathbb{R}; B_{p_0,1}^{s_2+\alpha_2}(\mathbb{R}))} \approx \|(2^{js_1} R^{j(s_2+\alpha_2)} a_j)_{j \geq 1}\|_{\ell^q} = \|(2^{js_1} 2^{\frac{\alpha_1}{\alpha_2} j s_2} 2^{j\alpha_1} a_j)_{j \geq 1}\|_{\ell^q}.$$

All the sequences in the above norms coincide. Hence the continuity of the embedding yields $\ell^q \hookrightarrow \ell^r$, which is false. \square

Remark 7.7. Let $r \in [1, q)$. As a consequence, in general it holds that

$$F_{p,q}^{s+\alpha}(\mathbb{R}; X) \cap F_{p,q}^s(\mathbb{R}; D(A)) \not\subseteq F_{p,r}^{s+\theta\alpha}(\mathbb{R}; D(A^{1-\theta}))$$

for a sectorial operator A (choose e.g. $A = -\Delta$ on $L^2(\mathbb{R}) = F_{2,2}^0(\mathbb{R})$).

8. TRACES OF WEIGHTED ANISOTROPIC SPACES OF INTERSECTION TYPE

Let A be a sectorial operator on a Banach space X and $w(t) = |t|^\gamma$ with $\gamma \in (-1, p-1)$. In this section we determine the image of the trace $\mathrm{tr}_0 u = u|_{t=0}$ for the spaces

$$F_{p,q}^{s+\alpha}(\mathbb{R}^d, w; X) \cap F_{p,q}^s(\mathbb{R}^d, w; D(A^\alpha)), \quad s < \frac{1+\gamma}{p} < s+\alpha,$$

and of similar spaces with F replaced by B , H or W . In particular, we will prove Theorem 1.4. We start with the continuity of the trace. Note that here we can cover the general case $\gamma > -1$.

Recall the following classical Hardy-Young inequality (see [23, p. 245-246]): for all measurable functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\alpha > 0$ and $p \in [1, \infty)$ it holds that

$$(8.1) \quad \int_0^\infty \sigma^{-\alpha p-1} \left(\int_0^\sigma f(t) dt \right)^p d\sigma \leq \alpha^{-p} \int_0^\infty \sigma^{p-\alpha p-1} f(\sigma)^p d\sigma.$$

For a short proof, set $u(\sigma) = \int_0^\sigma f(t) dt$ and note that $\sigma^{-\alpha-\frac{1}{p}} u(\sigma) = \int_0^1 \sigma^{1-\alpha-\frac{1}{p}} u'(\sigma\theta) d\theta$. Taking L^p -norms and applying Minkowski's inequality then gives (8.1).

Lemma 8.1. *Let X be a Banach space, $p \in (1, \infty)$, and $w(t) = |t|^\gamma$ with $\gamma > -1$. Suppose that $s \in \mathbb{R}$ and $\alpha > 0$ satisfy $s < \frac{1+\gamma}{p} < s+\alpha$, and let A be a sectorial operator on X with $\phi_A < \min\{\frac{\pi}{2}, \frac{\pi}{\alpha}\}$. Let $r \geq 0$ and $\theta = r + s + \alpha - \frac{1+\gamma}{p}$. Then the trace operator $\mathrm{tr}_0 u = u|_{t=0}$ maps continuously*

$$F_{p,q}^{s+\alpha}(\mathbb{R}, w; D(A^r)) \cap F_{p,q}^s(\mathbb{R}, w; D(A^{r+\alpha})) \rightarrow D_A(\theta, p), \quad q \in [1, \infty].$$

Proof. Without loss of generality we may assume that A is invertible. By the isomorphic properties of A^r (see Section 2.2), we may further assume that $r = 0$. In the first steps we consider $q = 1$. The extension to $q \in (1, \infty]$ will be considered at the end of the proof.

Step 1. First assume that

$$0 < s < \frac{1+\gamma}{p} < s+\alpha < 1.$$

Then an equivalent norm on $D_A(\theta, p)$ is given by (2.8). The idea of the following argument goes back to [17, Lemma 11] and [34, Lemma 4.1]. As in the proof of [36, Lemma 3.8] it can be seen that $\mathcal{S}(\mathbb{R}; D(A^\alpha))$ is a dense subset of $F_{p,1}^{s+\alpha}(\mathbb{R}, w; X) \cap F_{p,1}^s(\mathbb{R}, w; D(A^\alpha))$. For $u \in \mathcal{S}(\mathbb{R}; D(A^\alpha))$ the trace $\mathrm{tr}_0 u$ is given in a classical sense. Writing $u(t) - u(\tau) = \int_\tau^t u'(r) dr$, it is straightforward to check that for all $\sigma > 0$ we have

$$\mathrm{tr}_0 u = \sigma^{-1} \int_0^\sigma u(\tau) d\tau - \int_0^\sigma t^{-2} \int_0^t u(t) - u(\tau) d\tau dt.$$

This representation and (2.8) can be used to obtain

$$\|\mathrm{tr}_0 u\|_{D_A(\theta,p)} \leq C(T_1 + T_2),$$

where

$$\begin{aligned} T_1^p &= \int_0^\infty \sigma^{p(1-\theta)-1} \left\| \sigma^{-1} \int_0^\sigma A e^{-\sigma A} u(\tau) d\tau \right\|^p d\sigma, \\ T_2^p &= \int_0^\infty \sigma^{p(1-\theta)-1} \left\| \int_0^\sigma t^{-2} \int_0^t A e^{-\sigma A} (u(t) - u(\tau)) d\tau dt \right\|^p d\sigma. \end{aligned}$$

Note that $-1 < \gamma < p-1$ by the above assumption. We use $\|A^{1-\alpha} e^{-\sigma A}\|_{\mathcal{L}(X)} \leq C\sigma^{-1+\alpha}$ and Hölder's inequality to obtain

$$\begin{aligned} T_1^p &\leq \int_0^\infty \sigma^{-\theta p-1} \left(\int_0^\sigma \|A e^{-\sigma A} u(\tau)\| d\tau \right)^p d\sigma \\ &\leq \int_0^\infty \sigma^{-\theta p-1-(1-\alpha)p} \left(\int_0^\sigma \|u(\tau)\|_{D(A^\alpha)} d\tau \right)^p d\sigma \\ &\leq \int_0^\infty \sigma^{-\theta p-1-(1-\alpha)p} \left(\int_0^\sigma \tau^{-\frac{\gamma p'}{p}} d\tau \right)^{p/p'} \left(\int_0^\sigma \tau^\gamma \|u(\tau)\|_{D(A^\alpha)}^p d\tau \right) d\sigma \end{aligned}$$

$$\begin{aligned}
&= C \int_0^\infty \left(\int_\tau^\infty \sigma^{-sp-1} d\sigma \right) \tau^\gamma \|u(\tau)\|_{D(A^\alpha)}^p d\tau \\
&= C \int_0^\infty \tau^{\gamma-sp} \|u(\tau)\|_{D(A^\alpha)}^p d\tau \\
&\leq C \|u\|_{F_{p,1}^s(\mathbb{R}, w; D(A^\alpha))}^p.
\end{aligned}$$

In the last line we used $0 < s < \frac{1+\gamma}{p}$ and Hardy's inequality (Theorem 3.5).

To estimate T_2 , set $f(t) = t^{-2} \int_0^t \|u(t) - u(\tau)\| d\tau$. We use $\|Ae^{-\sigma A}\|_{\mathcal{L}(X)} \leq C\sigma^{-1}$, (8.1) and the equivalent norm from Proposition 6.1 for the F -spaces to obtain

$$\begin{aligned}
T_2^p &\leq C \int_0^\infty \sigma^{-\theta p-1} \left(\int_0^\sigma f(t) dt \right)^p d\sigma \\
&\leq C \int_0^\infty \sigma^{p-\theta p-1} f(\sigma)^p d\sigma \\
&= C \int_0^\infty \sigma^{-p-\theta p-1} \left(\int_0^\sigma \|u(\sigma) - u(\tau)\| d\tau \right)^p d\sigma \\
&= C \int_0^\infty \sigma^{-p-\theta p-1} \left(\int_{-\sigma}^0 \|u(\sigma) - u(h+\sigma)\| dh \right)^p d\sigma \\
&\leq C \int_0^\infty \sigma^{-p-\theta p-1} \sigma^{(s+\alpha+1)p} \left(\int_{-\sigma}^0 |h|^{-s-\alpha-1} \|u(\sigma) - u(h+\sigma)\| dh \right)^p d\sigma \\
&= C \int_0^\infty \sigma^\gamma \left(\int_{-\sigma}^0 |h|^{-s-\alpha-1} \|u(\sigma) - u(h+\sigma)\| dh \right)^p d\sigma \\
&\leq C [f]_{F_{p,1}^s(\mathbb{R}^d, w; X)}^{(1)} \leq C \|u\|_{F_{p,1}^{s+\alpha}(\mathbb{R}, w; X)}.
\end{aligned}$$

Therefore, we find

$$\|\mathrm{tr}_0 u\|_{D_A(\theta, p)} \leq C (\|u\|_{F_{p,1}^s(\mathbb{R}, w; D(A^\alpha))} + \|u\|_{F_{p,1}^{s+\alpha}(\mathbb{R}, w; X)})$$

for all $u \in \mathcal{S}(\mathbb{R}; D(A^\alpha))$, and the assertion follows from density.

Step 2. Assume $-1 < \gamma < p-1$ and $0 < s < \frac{1+\gamma}{p} < s+\alpha$. If $s+\alpha < 1$, then we are in the previous situation. Let $s+\alpha \geq 1$. Since $\frac{1+\gamma}{p} < 1$, we can find $\sigma \in (0, 1)$ such that $\frac{1+\gamma}{p} < s+\sigma\alpha < 1$. Set $\tilde{\alpha} = \sigma\alpha$ and $\tilde{r} = (1-\sigma)\alpha$. By assumption on ϕ_A , the operator A^α is still sectorial. We may thus apply Theorem 7.1 to obtain

$$F_{p,1}^{s+\alpha}(\mathbb{R}, w; X) \cap F_{p,1}^s(\mathbb{R}, w; D(A^\alpha)) \hookrightarrow F_{p,1}^{s+\tilde{\alpha}}(\mathbb{R}, w; D(A^{\tilde{r}})) \cap F_{p,1}^s(\mathbb{R}, w; D(A^{\tilde{r}+\tilde{\alpha}})).$$

Now it suffices to note that $\theta = \tilde{r} + s + \tilde{\alpha} - \frac{1+\gamma}{p}$ and to apply Step 1.

Step 3. Assume $-1 < \gamma < p-1$ and $s \leq 0 < \frac{1+\gamma}{p} < s+\alpha$. Then we find $\sigma \in (0, 1)$ such that $0 < s+\sigma\alpha < \frac{1+\gamma}{p}$. Setting $\tilde{s} = s+\sigma\alpha$ and $\tilde{\alpha} = (1-\sigma)\alpha$, as above Theorem 7.1 implies

$$F_{p,1}^{s+\alpha}(\mathbb{R}, w; X) \cap F_{p,1}^s(\mathbb{R}, w; D(A^\alpha)) \hookrightarrow F_{p,1}^{\tilde{s}+\tilde{\alpha}}(\mathbb{R}, w; X) \cap F_{p,1}^{\tilde{s}}(\mathbb{R}, w; D(A^{\tilde{\alpha}})).$$

Since $0 < \tilde{s} < \frac{1+\gamma}{p} < \tilde{s} + \tilde{\alpha}$ and $\theta = \tilde{s} + \tilde{\alpha} - \frac{1+\gamma}{p}$, we can apply Step 2.

Step 4. Assume $\gamma \geq p-1$, and that $s < \frac{1+\gamma}{p} < s+\alpha$ are arbitrary. Let $\tilde{s} = s - \frac{\gamma}{p}$. Then Theorem 3.3 gives

$$F_{p,1}^{s+\alpha}(\mathbb{R}, w; X) \cap F_{p,1}^s(\mathbb{R}, w; D(A^\alpha)) \hookrightarrow F_{p,1}^{\tilde{s}+\alpha}(\mathbb{R}; X) \cap F_{p,1}^{\tilde{s}}(\mathbb{R}; D(A^\alpha)).$$

Since $\tilde{s} < \frac{1}{p} < \tilde{s} + \alpha$ and $\theta = \tilde{s} + \alpha - \frac{1}{p}$, the result follows from the Steps 2 and 3.

Step 5. Finally we consider $q > 1$. For small $\varepsilon > 0$ we set

$$\tilde{s} = s - \frac{\varepsilon}{p}, \quad \tilde{\gamma} = \gamma - \varepsilon, \quad \tilde{w}(t) = |t|^{\tilde{\gamma}}.$$

Since $\frac{\gamma}{p} > \frac{\tilde{\gamma}}{p}$ and $s + \alpha - \frac{1+\gamma}{p} = \tilde{s} + \alpha - \frac{1+\tilde{\gamma}}{p}$, Theorem 3.3 implies

$$F_{p,q}^{s+\alpha}(\mathbb{R}, w; X) \hookrightarrow F_{p,1}^{\tilde{s}+\alpha}(\mathbb{R}, \tilde{w}; X), \quad F_{p,q}^s(\mathbb{R}, w; D(A^\alpha)) \hookrightarrow F_{p,1}^{\tilde{s}}(\mathbb{R}, \tilde{w}; D(A^\alpha)).$$

Now the continuity of tr_0 follows from the previous steps. \square

To obtain a continuous right-inverse for tr_0 we need some preparation. For $k \in \mathbb{N}_0$, $p \in (1, \infty)$ and $w \in A_p$ we set

$$W^{k,p}(\mathbb{R}_+^d, w; X) := \{f \in L^p(\mathbb{R}_+^d, w; X) : D^\alpha f \in L^p(\mathbb{R}_+^d, w; X) \text{ for every multiindex } |\alpha| \leq k\}.$$

Let $m \in \mathbb{N}$. A pointwise a.e. in \mathbb{R}_+^d defined linear map E is called an m -extension operator on \mathbb{R}_+^d , if for every Banach space X , $p \in (1, \infty)$, every $w \in A_p$ and every $0 \leq k \leq m$, E is bounded from $W^{k,p}(\mathbb{R}_+^d, w; X)$ into $W^{k,p}(\mathbb{R}^d, w; X)$, and for all $f \in W^{k,p}(\mathbb{R}_+^d, w; X)$ one has $(Ef)|_{\mathbb{R}_+^d} = f$.

Further, E is called a *total extension operator* on \mathbb{R}_+^d if it is an m -extension operator on \mathbb{R}_+^d for every $m \in \mathbb{N}$.

The following extensions of [2, Theorems 5.19 and 5.21] are straight forward.

Lemma 8.2.

- (1) For each $m \in \mathbb{N}_0$ there exists an m -extension operator E on \mathbb{R}_+^d . Moreover, for every $|\alpha| \leq m$, there exists an $m - |\alpha|$ -extension operator E_α such that for every Banach space X , $p \in (1, \infty)$, every $w \in A_p$ and $f \in W^{|\alpha|,p}(\mathbb{R}_+^d, w; X)$ it holds

$$D^\alpha Ef = E_\alpha D^\alpha f.$$

- (2) There exists a total extension operator \mathcal{E} from \mathbb{R}_+^d to \mathbb{R}^d such that for every Banach space X , $p \in (1, \infty)$, $k \in \mathbb{N}_0$ and $w(x', t) = |t|^\gamma$ with $\gamma \in (-1, p-1)$ we have

$$\mathcal{E} \in \mathcal{L}(W^{k,p}(\mathbb{R}_+^d, w; X), W^{k,p}(\mathbb{R}^d, w; X)).$$

Proof. Let $D = \{f|_{\mathbb{R}_+^d} : f \in C_c^\infty(\mathbb{R}^d; X)\}$. We claim that for any A_p -weight w and $k \in \mathbb{N}_0$, the space D is dense in $W^{k,p}(\mathbb{R}_+^d, w; X)$. To see this, let $f \in W^{k,p}(\mathbb{R}_+^d, w; X)$ and extend it by zero to \mathbb{R}^d . Let $\phi \in C_c^\infty(\mathbb{R}^d; X)$ be such that $\phi(x', t) = 0$ for $t \geq 0$, $\phi \geq 0$ and $\int \phi dx = 1$. Let $\phi_n(x) = n^d \phi(nx)$ and $f_n = \phi_n * f$. For each $|\alpha| \leq k$, let $g_\alpha = D^\alpha f$ and extend it by zero to \mathbb{R}^d . One has $D^\alpha f_n = \phi_n * g_\alpha$ on \mathbb{R}_+^d . Now it follows from [53, Theorem 2.1.4] that $\lim_{n \rightarrow \infty} f_n = f$ in $W^{k,p}(\mathbb{R}_+^d, w; X)$. Moreover, each f_n belongs to $C^\infty(\mathbb{R}^d; X)$. In order to find an approximation with elements in D it remains to multiply each f_n by a smooth cut-off function $\zeta_n \in C_c^\infty(\mathbb{R}^d)$ which satisfies $\zeta_n = 1$ on $B(0, R_n)$ for R_n large enough.

Given $m \in \mathbb{N}_0$, the extension operators E and E_α from the proof of [2, Theorem 5.19] are bounded from D to $W^{k,p}(\mathbb{R}^d, w; X)$ for every $k \leq m$ and $k \leq m - |\alpha|$, respectively. The total extension operator \mathcal{E} from the proof of [2, Theorem 5.21] is bounded from D to $W^{m,p}(\mathbb{R}^d, w; X)$ for every $m \in \mathbb{N}_0$. Now the assertions follow from the density of D . \square

The following lemma is the main tool for the right-inverse of tr_0 .

Lemma 8.3. Let A be a sectorial operator on X with $\phi_A < \frac{\pi}{2}$. Let $p \in (1, \infty)$, $w(t) = |t|^\gamma$ with $-1 < \gamma < p-1$ and $\beta > \frac{1+\gamma}{p}$. Let E be either a total extension operator or an m -extension operator with $m \geq \beta + 1$ from Lemma 8.2 for \mathbb{R}_+ . Then $Ee^{-(1+A)}$ maps continuously

$$D_A\left(\beta - \frac{1+\gamma}{p}, p\right) \rightarrow F_{p,q}^{\theta\beta}(\mathbb{R}, w; D(A^{(1-\theta)\beta})), \quad q \in [1, \infty], \quad \theta \in [0, 1].$$

If A is exponentially stable, then one can replace $1 + A$ by A in the above lemma.

Proof. Replacing A by $1 + A$ if necessary, we may assume A is exponentially stable. To prove the result it is sufficient to consider $q = 1$. Throughout, let $\theta \in [0, 1]$.

Step 1. Assume there is $k \in \mathbb{N}$ such that $k - 1 + \frac{1+\gamma}{p} < \theta\beta < k$. Using (2.8), we estimate

$$\begin{aligned} \|Ee^{-A}x\|_{W^{k,p}(\mathbb{R}, |\cdot|^{\gamma+p(k-\theta\beta)}; D(A^{(1-\theta)\beta}))} &\leq C\|e^{-A}x\|_{W^{k,p}(\mathbb{R}_+, |\cdot|^{\gamma+p(k-\theta\beta)}; D(A^{(1-\theta)\beta}))} \\ &\leq C\|Ae^{-A}x\|_{L^p(\mathbb{R}_+, |\cdot|^{\gamma+p(k-\theta\beta)}; D(A^{k-1+(1-\theta)\beta}))} \\ &\leq C\|A^{k-1+(1-\theta)\beta}x\|_{D_A(\theta\beta-k+1-\frac{1+\gamma}{p}, p)} \\ &= C\|x\|_{D_A(\beta-\frac{1+\gamma}{p}, p)}. \end{aligned}$$

Since $\gamma + p(k - \theta\beta) < p - 1$, (2.5) and Theorem 3.3 imply that

$$\begin{aligned} W^{k,p}(\mathbb{R}, |\cdot|^{\gamma+p(k-\theta\beta)}; D(A^{(1-\theta)\beta})) &\hookrightarrow F_{p,\infty}^k(\mathbb{R}, |\cdot|^{\gamma+p(k-\theta\beta)}; D(A^{(1-\theta)\beta})) \\ &\hookrightarrow F_{p,1}^{\theta\beta}(\mathbb{R}, w; D(A^{(1-\theta)\beta})), \end{aligned}$$

which shows the asserted mapping property.

Step 2. Assume $0 \leq \theta\beta \leq \frac{1+\gamma}{p}$. As before it follows from (2.8) and $\gamma < p - 1$ that

$$\|Ee^{-\cdot A}x\|_{F_{p,\infty}^0(\mathbb{R}, w; D(A^\beta))} \leq C\|Ee^{-\cdot A}x\|_{L^p(\mathbb{R}, w; D(A^\beta))} \leq C\|x\|_{D_A(\beta - \frac{1+\gamma}{p}, p)}.$$

Choose $\varepsilon > 0$ such that $\frac{1+\gamma}{p} + \varepsilon < \min\{\beta, 1\}$. Using Step 1, it follows that $Ee^{-\cdot A}$ maps $D_A(\beta - \frac{1+\gamma}{p}, p)$ into

$$(8.2) \quad F_{p,\infty}^{\frac{1+\gamma}{p} + \varepsilon}(\mathbb{R}, w; D(A^{\beta - \frac{1+\gamma}{p} - \varepsilon})) \cap F_{p,\infty}^0(\mathbb{R}, w; D(A^\beta)).$$

Since $\frac{1+\gamma}{p} + \varepsilon < 1$, the operator $A^{\frac{1+\gamma}{p} + \varepsilon}$ is sectorial. Thus Theorem 7.1 applies and (8.2) embeds into

$$F_{p,\infty}^{\theta\beta}(\mathbb{R}, w; D(A^{(1-\theta)\beta})).$$

We improve this mapping property of $Ee^{-\cdot A}$ as follows. For small $\varepsilon > 0$ we set $\tilde{\beta} = \beta + \varepsilon/p$, $\tilde{\theta} = \frac{\theta\beta + \varepsilon/p}{\beta + \varepsilon/p}$ and $\tilde{\gamma} = \gamma + \varepsilon$, such that $\tilde{\theta}\tilde{\beta} \leq \frac{1+\tilde{\gamma}}{p}$ and let $\tilde{w}(x) = |x|^{\tilde{\gamma}}$. Then by the above considerations and Theorem 3.3, the operator $Ee^{-\cdot A}$ maps $D_A(\beta - \frac{1+\gamma}{p}, p) = D_A(\tilde{\beta} - \frac{1+\tilde{\gamma}}{p}, p)$ into

$$F_{p,\infty}^{\tilde{\theta}\tilde{\beta}}(\mathbb{R}, \tilde{w}; D(A^{(1-\tilde{\theta})\tilde{\beta}})) = F_{p,\infty}^{\theta\beta + \varepsilon/p}(\mathbb{R}, \tilde{w}; D(A^{(1-\theta)\beta})) \hookrightarrow F_{p,1}^{\theta\beta}(\mathbb{R}, w; D(A^{(1-\theta)\beta})).$$

Step 3. It remains to consider the case $k \leq \theta\beta \leq k + \frac{1+\gamma}{p}$ for $k \in \mathbb{N}$. In this step we first assume that there is $k_0 \in \mathbb{N}$ such that $k_0 + \frac{1+\gamma}{p} < \beta < k_0$. Then there are $\theta_1, \theta_2 \in (0, 1)$ such that $\theta_1 < \theta < \theta_2$ and $k - 1 + \frac{1+\gamma}{p} < \theta_1\beta < k$ and $k + \frac{1+\gamma}{p} < \theta_2\beta < k + 1$. By Step 1, the operator $Ee^{-\cdot A}$ maps $D_A(\beta - \frac{1+\gamma}{p}, p)$ into

$$(8.3) \quad F_{p,1}^{\theta_2\beta}(\mathbb{R}, w; D(A^{(1-\theta_2)\beta})) \cap F_{p,1}^{\theta_1\beta}(\mathbb{R}, w; D(A^{(1-\theta_1)\beta})).$$

Since $(1 - \theta_1)\beta - (1 - \theta_2)\beta = (\theta_2 - \theta_1)\beta \in (0, 2)$, Theorem 7.1 applies to the sectorial operator $A^{(\theta_2 - \theta_1)\beta}$ on $D(A^{(1-\theta_2)\beta})$, and thus (8.3) embeds into $F_{p,1}^{\theta\beta}(\mathbb{R}, w; D(A^{(1-\theta)\beta}))$.

Step 4. Let again $k \leq \theta\beta \leq k + \frac{1+\gamma}{p}$ for some $k \in \mathbb{N}$, and now assume that there is $k_0 \in \mathbb{N}$ such that $k_0 \leq \beta \leq k_0 + \frac{1+\gamma}{p}$. Take $\tilde{\beta} \in (0, 1)$ such that $k_0 + \frac{1+\gamma}{p} < \beta + \tilde{\beta} < k_0 + 1$. Let $\tau \in [0, 1]$. Applying the previous steps with β replaced by $\beta + \tilde{\beta}$, for any $\tau \in [0, 1]$ and $y \in D_A(\beta + \tilde{\beta} - \frac{1+\gamma}{p}, p)$ we get that

$$\|Ee^{-\cdot A}y\|_{F_{p,1}^{\tau(\beta + \tilde{\beta})}(\mathbb{R}, w; D(A^{(1-\tau)(\beta + \tilde{\beta})}))} \leq C\|y\|_{D_A(\beta + \tilde{\beta} - \frac{1+\gamma}{p}, p)}.$$

Choosing $\tau = \frac{\theta\beta}{\beta + \tilde{\beta}}$ leads to

$$\|Ee^{-\cdot A}y\|_{F_{p,1}^{\theta\beta}(\mathbb{R}, w; D(A^{(1-\theta)\beta + \tilde{\beta}}))} \leq C\|y\|_{D_A(\beta + \tilde{\beta} - \frac{1+\gamma}{p}, p)}.$$

For $x \in D_A(\beta - \frac{1+\gamma}{p}, p)$ we may now set $y = A^{-\tilde{\beta}}x$ to obtain

$$\|Ee^{-\cdot A}x\|_{F_{p,1}^{\theta\beta}(\mathbb{R}, w; D(A^{(1-\theta)\beta}))} \leq C\|x\|_{D_A(\beta - \frac{1+\gamma}{p}, p)}.$$

□

This result can be applied to the intersection spaces under consideration as follows.

Lemma 8.4. *Let A be a sectorial operator on a Banach space X with $\phi_A < \frac{\pi}{2}$ and $r \geq 0$. Let $p \in (1, \infty)$, $w(t) = |t|^\gamma$ with $\gamma \in (-1, p - 1)$, $s \in \mathbb{R}$ and $\alpha > 0$ be such that $s < \frac{1+\gamma}{p} < s + \alpha$.*

(1) *There is an m -extension operator E with $m \geq |s| + \alpha + 1$ such that $Ee^{-\cdot(1+A)}$ maps continuously*

$$D_A\left(r + s + \alpha - \frac{1+\gamma}{p}, p\right) \rightarrow F_{p,q}^{s+\theta\alpha}(\mathbb{R}, w; D(A^{r+(1-\theta)\alpha})), \quad q \in [1, \infty], \quad \theta \in [0, 1].$$

(2) Let \mathcal{E} be the total extension operator from Lemma 8.2 and assume additionally that $s > \frac{1+\gamma}{p} - 1$. Then $\mathcal{E}e^{-(1+A)}$ maps continuously

$$D_A\left(r + s + \alpha - \frac{1+\gamma}{p}, p\right) \rightarrow F_{p,q}^{s+\theta\alpha}(\mathbb{R}, w; D(A^{r+(1-\theta)\alpha})), \quad q \in [1, \infty], \quad \theta \in [0, 1].$$

Proof. We may assume that A is exponentially stable. To prove the result it is sufficient to consider $q = 1$. Moreover, it is enough to consider $r = 0$.

Step 1. Let E be any of the extension operators as above. If $s \geq 0$, then Lemma 8.3 with $\beta = s + \alpha$ and $\theta' = \frac{s+\theta\alpha}{s+\alpha}$ yields both assertions. Next assume $\frac{\gamma+1}{p} - 1 < s < 0$, such that $-1 < \gamma - sp < p - 1$. First, Lemma 8.3 shows the assertion for $\theta \in [-s/\alpha, 1]$. Observe that here any $s < 0$ is allowed. In particular, for all $s < 0$ one has that

$$Ee^{-A} : D_A\left(s + \alpha - \frac{1+\gamma}{p}, p\right) \rightarrow F_{p,q}^0(\mathbb{R}, w; D(A^{s+\alpha})).$$

For the remaining values of θ , let $v(t) = |t|^{\gamma-sp}$. Since $D_A\left(s + \alpha - \frac{1+\gamma}{p}, p\right) = D_A\left(\alpha - \frac{1+\gamma-sp}{p}, p\right)$, it follows from Lemma 8.3 with $\beta = \alpha$, weight v and $\theta = 0$ that

$$Ee^{-A} : D_A\left(s + \alpha - \frac{1+\gamma}{p}, p\right) \rightarrow F_{p,q}^0(\mathbb{R}, v; D(A^\alpha)),$$

is bounded. From Theorem 3.3 one sees that

$$F_{p,q}^0(\mathbb{R}, v; D(A^\alpha)) \hookrightarrow F_{p,q}^s(\mathbb{R}, w; D(A^\alpha)).$$

This shows the assertion for $\theta = 0$. For $\theta \in (0, -s/\alpha)$ we may apply Theorem 7.1 to the sectorial operator A^{-s} to obtain

$$F_{p,q}^0(\mathbb{R}, w; D(A^{s+\alpha})) \cap F_{p,q}^s(\mathbb{R}, w; D(A^\alpha)) \hookrightarrow F_{p,q}^{s+\theta\alpha}(\mathbb{R}, w; D(A^{(1-\theta)\alpha})).$$

Now (1) and (2) follow in case $s > \frac{\gamma+1}{p} - 1$.

Step 2. Assume that $s \leq \frac{\gamma+1}{p} - 1$. The case $\theta \in [-s/\alpha, 1]$ was treated before. Let $\theta \in [0, -s/\alpha]$. Since $s + \theta\alpha < 0$ we can find $k \in \mathbb{N}_0$ such that $\frac{\gamma+1}{p} - 1 < s + k + \theta\alpha \leq \frac{\gamma+1}{p}$. Note that $(1-\theta)\alpha - k = s + \alpha - (s+k) - \theta\alpha > 0$. By Lemma 8.2 there is an $(m+k)$ -extension operator E_m and an m -extension operator E such that $\partial_t^k E_m f = E \partial_t^k f$. Using [36, Proposition 3.10], we obtain

$$\begin{aligned} \|Ee^{-A}x\|_{F_{p,q}^{s+\theta\alpha}(\mathbb{R}, w; D(A^{(1-\theta)\alpha}))} &= \|E\partial_t^k e^{-A}x\|_{F_{p,q}^{s+\theta\alpha}(\mathbb{R}, w; D(A^{(1-\theta)\alpha-k}))} \\ &= \|\partial_t^k E_m e^{-A}x\|_{F_{p,q}^{s+\theta\alpha}(\mathbb{R}, w; D(A^{(1-\theta)\alpha-k}))} \\ &\leq C\|E_m e^{-A}x\|_{F_{p,q}^{s+k+\theta\alpha}(\mathbb{R}, w; D(A^{(1-\theta)\alpha-k}))}. \end{aligned}$$

Now Step 1 applies to E_m , $\tilde{s} = s + \theta\alpha + k - \varepsilon$, $\tilde{\alpha} = (1-\theta)\alpha - k + \varepsilon$ and parameter $\sigma = \frac{\varepsilon}{\alpha}$ for some small $\varepsilon > 0$, which yields

$$\begin{aligned} \|E_m e^{-A}x\|_{F_{p,q}^{s+k}(\mathbb{R}, w; D(A^{\alpha-k}))} &= \|E_m e^{-A}x\|_{F_{p,q}^{\tilde{s}+\sigma\tilde{\alpha}}(\mathbb{R}, w; D(A^{(1-\sigma)\tilde{\alpha}}))} \\ &\leq C\|x\|_{D_A(\tilde{s}+\tilde{\alpha}-\frac{1+\gamma}{p}, p)} = C\|x\|_{D_A(s+\alpha-\frac{1+\gamma}{p}, p)}. \end{aligned}$$

□

Proof of Theorem 1.4: First consider the case of F -spaces. The continuity of the trace follows from Lemma 8.1. The operator constructed in Lemma 8.4 gives the required right-inverse. The case of B -spaces is included in the previous argument since $B_{p,p}^s = F_{p,p}^s$. Moreover, the assertions for H - and W -spaces follow from (2.4). □

9. THE L^p - L^q TWO PHASE STEFAN PROBLEM WITH GIBBS-THOMSON CORRECTION

As an application of our embedding and trace results we show maximal L^p - L^q -regularity for the fully inhomogeneous linearized two phase Stefan problem with Gibbs-Thomson correction, which is given by

$$(9.1) \quad \begin{cases} \mu u + \partial_t u - \Delta u &= f & \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^d, \\ \llbracket u \rrbracket &= 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}^{d-1} \\ u + \Delta' h &= g_1 & \text{on } \mathbb{R}_+ \times \mathbb{R}^{d-1}, \\ \partial_t h - \llbracket \partial_\nu u \rrbracket &= g_2 & \text{on } \mathbb{R}_+ \times \mathbb{R}^{d-1} \\ u|_{t=0} &= u_0 & \text{in } \dot{\mathbb{R}}^d, \\ h|_{t=0} &= h_0 & \text{on } \mathbb{R}^{d-1}. \end{cases}$$

Here we have $\mu > 0$, $\dot{\mathbb{R}}^d = \mathbb{R}^{d-1} \times (\mathbb{R} \setminus \{0\})$, and

$$\llbracket u \rrbracket := u|_{\mathbb{R}_+^d} - u|_{\mathbb{R}_-^d}, \quad \llbracket \partial_\nu u \rrbracket := \partial_{x_d} u|_{\mathbb{R}_-^d} - \partial_{x_d} u|_{\mathbb{R}_+^d}$$

denote the jump of u and of the outer normal derivative $\partial_\nu u$ along $\partial \dot{\mathbb{R}}^d = \mathbb{R}^{d-1}$, respectively. Moreover, Δ is the Laplacian on \mathbb{R}^d and Δ' is the Laplacian on \mathbb{R}^{d-1} . Note that the equation in the third line of (9.1) must be read as $u|_{\mathbb{R}^{d-1}} + \Delta' h = g_1$. The inhomogeneities f, g_1, g_2 and the initial values u_0, h_0 are assumed to be given. We are looking for strong solutions (u, h) which satisfy (9.1) pointwise almost everywhere.

One ends up with (9.1) after transforming and linearizing the full two-phase Stefan problem with Gibbs-Thomson correction, which is a free boundary problem modelling phase transitions in liquid-solid systems, to a fixed phase boundary and extending to $\dot{\mathbb{R}}^d$, see [18, Section 7]. The unknown u describes the heat concentration in the phases \mathbb{R}_\pm^d , and the graph of the second unknown h , which only lives on the boundary, is the transformed free phase boundary.

The corresponding one phase problem was considered in an L^p - L^p -setting in [16, Section 5]. The fully inhomogeneous two phase problem (9.1) was treated in [18, Theorem 6.1] in an L^p - L^p -setting. For trivial initial data $u_0 = 0$ and $h_0 = 0$ it was treated in an L^p - L^q -setting in [28, Theorem 7.34], where $p \in (1, \infty)$ and $\frac{2p}{p+1} < q < 2p$.

In the L^p - L^q -approach Triebel-Lizorkin spaces naturally come into play for $p \neq q$ as the optimal time regularity of the boundary inhomogeneities and the unknown h . We also refer to [55, 14] for the case of a heat equation with inhomogeneous Dirichlet or Neumann boundary conditions. In general, the motivation to work in an L^p - L^q -setting is when the scaling of the basic underlying space $L^p(\mathbb{R}_+; L^q(\dot{\mathbb{R}}^d))$ fits to the scaling of the problem under consideration only if $p \neq q$ (see e.g. [47, Section 1]).

The purpose of this section is to extend the maximal L^p - L^q -regularity results of [28, Theorem 7.34] for (9.1) to the case of nontrivial initial values u_0 and h_0 .

In the L^p - L^q -approach one starts with

$$f \in \mathbb{E}_0 := L^p(\mathbb{R}_+; L^q(\dot{\mathbb{R}}^d)), \quad p, q \in (1, \infty),$$

and looks for a solution (u, h) such that, at first,

$$u \in \mathbb{E}_u := H^{1,p}(\mathbb{R}_+; L^q(\dot{\mathbb{R}}^d)) \cap L^p(\mathbb{R}_+; H^{2,q}(\dot{\mathbb{R}}^d)).$$

The boundary inhomogeneities g_1 and g_2 should have at least the regularity of the terms involving u arising in the corresponding equations. Combining [27, Theorem 2.2] and [28, Proposition 5.23] (see also [14, Proposition 6.4]), this suggests that

$$g_1 \in \mathbb{F}_1 := F_{p,q}^{1-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{2-1/q}(\mathbb{R}^{d-1})),$$

$$g_2 \in \mathbb{F}_2 := F_{p,q}^{1/2-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{1-1/q}(\mathbb{R}^{d-1})).$$

Here, for a Banach space X , $s > 0$ and $p, q \in (1, \infty)$, the Triebel-Lizorkin spaces over the half-line are defined by restriction (see [28, Definition 5.4]), i.e.,

$$F_{p,q}^s(\mathbb{R}_+; X) := \{f \in L^p(\mathbb{R}_+; X) : \exists g \in F_{p,q}^s(\mathbb{R}; X) \text{ with } g|_{\mathbb{R}_+} = f\}.$$

It is shown in [28, Corollary 5.12] that for all $s > 0$, $r \in \mathbb{R}$ and $\frac{2p}{p+1} < q < 2p$ the restriction from \mathbb{R} to \mathbb{R}_+ is a retraction from $F_{p,q}^s(\mathbb{R}; H^{r,q}(\mathbb{R}^{d-1}))$ to $F_{p,q}^s(\mathbb{R}_+; H^{r,q}(\mathbb{R}^{d-1}))$, and that there exists a universal coretraction. Of course, the same is true for any vector-valued L^p -space. This allows to transfer results for F -spaces over \mathbb{R} to the corresponding F -spaces over \mathbb{R}_+ by an extension-restriction argument. Here and below, the condition $\frac{2p}{p+1} < q < 2p$ imposed [28] should not be essential, see also Remark 9.8. Note that in particular the case $p = q$ is covered.

The regularity of the boundary unknown h should now be such that $\Delta' h \in \mathbb{F}_1$ and $\partial_t h \in \mathbb{F}_2$. We claim that

$$h \in \mathbb{E}_h := F_{p,q}^{3/2-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap F_{p,q}^{1-1/(2q)}(\mathbb{R}_+; H^{2,q}(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{4-1/q}(\mathbb{R}^{d-1}))$$

is sufficient for this purpose.

Lemma 9.1. *For $p, q \in (1, \infty)$ define*

$$\tilde{\mathbb{E}}_h := F_{p,q}^{3/2-1/(2q)}(\mathbb{R}; L^q(\mathbb{R}^{d-1})) \cap F_{p,q}^{1-1/(2q)}(\mathbb{R}; H^{2,q}(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}; B_{q,q}^{4-1/q}(\mathbb{R}^{d-1})),$$

and analogously $\tilde{\mathbb{F}}_1$ and $\tilde{\mathbb{F}}_2$. Then $\Delta' : \tilde{\mathbb{E}}_h \rightarrow \tilde{\mathbb{F}}_1$ and $\partial_t : \tilde{\mathbb{E}}_h \rightarrow \tilde{\mathbb{F}}_2$ are continuous.

Proof. The assertion for Δ' follows from a direct pointwise estimate in the F - and the L^p -norm. For the continuity of ∂_t , we use Theorem 7.1 to obtain

$$\tilde{\mathbb{E}}_h \hookrightarrow F_{p,q}^{3/2-1/(2q)}(\mathbb{R}; L^q(\mathbb{R}^{d-1})) \cap F_{p,q}^{1-1/(2q)}(\mathbb{R}; H^{2,q}(\mathbb{R}^{d-1})) \hookrightarrow F_{p,q}^{1+\varepsilon}(\mathbb{R}; H^{2-2/q-4\varepsilon,q}(\mathbb{R}^{d-1})),$$

where $\varepsilon > 0$ is such that $2 - 2/q - 4\varepsilon > 1 - 1/q$. Using that then $F_{p,q}^{1+\varepsilon} \hookrightarrow H^{1,p}$ and $H^{2-2/q-4\varepsilon,q} \hookrightarrow B_{q,q}^{1-1/q}$, we get

$$\tilde{\mathbb{E}}_h \hookrightarrow F_{p,q}^{3/2-1/(2q)}(\mathbb{R}; L^q(\mathbb{R}^{d-1})) \cap H^{1,p}(\mathbb{R}; B_{q,q}^{1-1/q}(\mathbb{R}^{d-1})).$$

This embedding together with [36, Proposition 3.10] shows that $\partial_t : \tilde{\mathbb{E}}_h \rightarrow \tilde{\mathbb{F}}_2$ is continuous. \square

Remark 9.2. Employing [28, Corollary 5.12] and an extension-restriction argument, for $\frac{2p}{p+1} < q < 2p$ the assertions of Lemma 9.1 remain true if one replaces $\tilde{\mathbb{E}}_h$, $\tilde{\mathbb{F}}_1$, $\tilde{\mathbb{F}}_2$ by \mathbb{E}_h , \mathbb{F}_1 , \mathbb{F}_2 .

In the following we determine the temporal trace spaces of \mathbb{E}_u and \mathbb{E}_h , to which $u_0 = u|_{t=0}$ and $h_0 = h|_{t=0}$ necessarily belong if $(u, h) \in \mathbb{E}_u \times \mathbb{E}_h$.

For $s > 0$, let $B_{q,p}^s(\mathbb{R}^d)$ be the space of all $v_0 \in L^q(\mathbb{R}^d)$ such that $v_0|_{\mathbb{R}_\pm^d} \in B_{q,p}^s(\mathbb{R}_\pm^d)$, where the latter spaces are as above defined by restriction (see [50, Section 4.2.1]). It then follows from an extension-restriction argument and Theorem 1.4 that

$$\text{tr}_0 u := u|_{t=0}$$

maps \mathbb{E}_u continuously onto

$$X_u := B_{q,p}^{2-2/p}(\mathbb{R}^d).$$

The temporal trace space of \mathbb{E}_h will be deduced from the following result.

Proposition 9.3. *Let $p, q \in (1, \infty)$ be such that $1 - 1/(2q), 1/2 - 1/(2q) \neq 1/p$ and consider again*

$$\tilde{\mathbb{E}}_h = F_{p,q}^{3/2-1/(2q)}(\mathbb{R}; L^q(\mathbb{R}^{d-1})) \cap F_{p,q}^{1-1/(2q)}(\mathbb{R}; H^{2,q}(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}; B_{q,q}^{4-1/q}(\mathbb{R}^{d-1})).$$

Then tr_0 maps $\tilde{\mathbb{E}}_h$ continuously onto X_h , where

$$X_h := B_{q,p}^{6-2/q-4/p}(\mathbb{R}^{d-1}) \quad \text{if } 1 - 1/(2q) < 1/p, \quad X_h := B_{q,p}^{4-1/q-2/p}(\mathbb{R}^{d-1}) \quad \text{if } 1 - 1/(2q) > 1/p.$$

Moreover, if $1/2 - 1/(2q) > 1/p$, then the operator $\text{tr}_0 \partial_t$ maps $\tilde{\mathbb{E}}_h$ continuously onto

$$X_{\partial_t h} := B_{q,p}^{2-2/q-4/p}(\mathbb{R}^{d-1}).$$

In each case there is a continuous map $\mathcal{R} : X_h \times X_{\partial_t h} \rightarrow \tilde{\mathbb{E}}_h$ such that $\text{tr}_0 \mathcal{R}(h_0, h_1) = h_0$ and, in case $1/2 - 1/(2q) > 1/p$, such that $\text{tr}_0 \partial_t \mathcal{R}(h_0, h_1) = h_1$ for all $(h_0, h_1) \in X_h \times X_{\partial_t h}$.

Proof. To economize the notation we write $L^q = L^q(\mathbb{R}^{d-1})$, $F_{q,1}^s = F_{q,1}^s(\mathbb{R}^{d-1})$ and so on. Throughout, let \mathcal{E} be the total extension operator from Lemma 8.2.

Step 1. Assume $1 - 1/(2q) < 1/p$. Let $X = L^q$ and $A = (1 - \Delta')^2$ with $D(A) = H^{4,q}$. Then Theorem 1.4 implies that tr_0 maps

$$\tilde{\mathbb{E}}_h \hookrightarrow F_{p,q}^{3/2-1/(2q)}(\mathbb{R}; X) \cap F_{p,q}^{1-1/(2q)}(\mathbb{R}; D(A^{1/2}))$$

into $D_A(3/2 - 1/(2q) - 1/p, p) = B_{q,p}^{6-2/q-4/p}$. For the right-inverse, consider $A = (1 - \Delta')^2$ on $X = F_{q,1}^0$ with $D(A) = F_{q,1}^4$. By Corollary 5.4 the operator A is sectorial with angle $\phi_A = 0$. Then it follows from [50, Theorem 2.4.2.1] that $B_{q,p}^{6-2/q-4/p} = D_A(3/2 - 1/(2q) - 1/p, p)$ as well. By Lemma 8.4, the operator $\mathcal{R}(h_0, h_1) := \mathcal{E}e^{-\cdot A}h_0$ maps this space continuously into

$$\mathbb{Y} := F_{p,1}^{3/2-1/(2q)}(\mathbb{R}; F_{q,1}^0) \cap F_{p,1}^0(\mathbb{R}; F_{q,1}^{6-2/q}).$$

Since (2.4) implies that $F_{q,1}^0 \hookrightarrow L^q$ and $F_{q,1}^{6-2/q} \hookrightarrow B_{q,q}^{4-1/q}$, it remains to show that \mathbb{Y} embeds into $F_{p,q}^{1-1/(2q)}(\mathbb{R}; H^{2,q})$. But this is a consequence of Theorem 7.1 and (2.4).

Step 2. Assume $1 - 1/(2q) > 1/p > 1/2 - 1/(2q)$. As above we start with the continuity of the trace. First we have

$$F_{p,q}^{1-1/(2q)}(\mathbb{R}; H^{2,q}) \hookrightarrow F_{p,\infty}^{1-1/(2q)}(\mathbb{R}; F_{q,\infty}^2), \quad L^p(\mathbb{R}; B_{q,q}^{4-1/q}) \hookrightarrow F_{p,\infty}^0(\mathbb{R}; F_{q,\infty}^{4-1/q}).$$

Considering $A = 1 - \Delta'$ on $X = F_{q,\infty}^2$ with $D(A) = F_{q,\infty}^4$, it follows from Theorem 1.4 and interpolation that tr_0 maps $\tilde{\mathbb{E}}_h$ continuously into

$$(X, D(A))_{1-1/(2q)-1/p, p} = B_{q,p}^{4-1/q-2/p}.$$

For the right-inverse we consider $A = 1 - \Delta'$ on $X = F_{q,1}^0$ with $D(A) = F_{q,1}^2$. Applying Lemma 8.4, we get that $\mathcal{R}(h_0, h_1) := \mathcal{E}e^{-\cdot A}h_0$ maps $B_{q,p}^{4-1/q-2/p} = D_A(2 - 1/(2q) - 1/p, p)$ continuously into

$$\mathbb{Y} := F_{p,1}^{2-1/(2q)}(\mathbb{R}; F_{q,1}^0) \cap F_{p,1}^0(\mathbb{R}; F_{q,1}^{4-1/q}).$$

As above we obtain from (2.4) that \mathbb{Y} embeds into $F_{p,q}^{3/2-1/(2q)}(\mathbb{R}; L^q)$ and $L^p(\mathbb{R}; B_{q,q}^{4-1/q})$, and Theorem 7.1 shows that $\mathbb{Y} \hookrightarrow F_{p,q}^{1-1/(2q)}(\mathbb{R}; H^{2,q})$.

Step 3. Assume $1/2 - 1/(2q) > 1/p$. For $h \in \tilde{\mathbb{E}}_h$, the regularity of $\text{tr}_0 h$ is obtained as in the previous step. We determine the regularity of $\text{tr}_0 \partial_t h$. Theorem 7.1 gives

$$\tilde{\mathbb{E}}_h \hookrightarrow F_{p,q}^{3/2-1/(2q)}(\mathbb{R}; L^q) \cap F_{p,q}^{1-1/(2q)}(\mathbb{R}; H^{2,q}) \hookrightarrow F_{p,q}^1(\mathbb{R}; H^{2-2/q,q}).$$

It thus follows from [36, Proposition 3.10] that

$$\partial_t : \tilde{\mathbb{E}}_h \rightarrow F_{p,q}^{1/2-1/(2q)}(\mathbb{R}; L^q) \cap F_{p,q}^0(\mathbb{R}; H^{2-2/q,q})$$

is continuous. Applying Theorem 1.4 with $A = (1 - \Delta')^2$ on $X = L^q$, we get that

$$\text{tr}_0 \partial_t : \tilde{\mathbb{E}}_h \rightarrow (L^q, H^{4,q})_{1/2-1/(2q)-1/p, p} = B_{q,p}^{2-2/q-4/p}$$

is continuous. For the right-inverse we let $X = F_{q,1}^0$, $A_0 = 1 - \Delta'$ with $D(A_0) = F_{q,1}^2$ and $A_1 = (1 - \Delta')^2$ with $D(A_1) = F_{q,1}^4$. Following the considerations in [15, Section 4.1], we set

$$\mathcal{R}_0 := 2\mathcal{E}e^{-\cdot A_0} - \mathcal{E}e^{-\cdot 2A_0}, \quad \mathcal{R}_1 := (\mathcal{E}e^{-\cdot A_1} - \mathcal{E}e^{-\cdot 2A_1})A_1^{-1}.$$

Note that $\text{tr}_0 \mathcal{R}_0 = \text{id}$, $\text{tr}_0 \partial_t \mathcal{R}_0 = 0$, $\text{tr}_0 \mathcal{R}_1 = 0$ and $\text{tr}_0 \partial_t \mathcal{R}_1 = \text{id}$. Hence

$$\mathcal{R}(h_0, h_1) := \mathcal{R}_0 h_0 + \mathcal{R}_1 h_1$$

satisfies $\text{tr}_0 \mathcal{R}(h_0, h_1) = h_0$ and $\text{tr}_0 \partial_t \mathcal{R}(h_0, h_1) = h_1$. In Step 2 we have shown that $\mathcal{R}_0 : X_h \rightarrow \tilde{\mathbb{E}}_h$ is continuous. Moreover, since A_1^{-1} maps $X_{\partial_t h}$ to $B_{q,p}^{6-2/q-4/p}$ and since we have shown in Step 1 that $\mathcal{E}e^{-\cdot A_1}$ maps $B_{q,p}^{6-2/q-4/p}$ to $\tilde{\mathbb{E}}_h$, we obtain that $\mathcal{R}_1 : X_{\partial_t h} \rightarrow \tilde{\mathbb{E}}_h$ is continuous. Therefore $\mathcal{R} : X_h \times X_{\partial_t h} \rightarrow \tilde{\mathbb{E}}_h$ is continuous as well. \square

Remark 9.4. The methods from the proof above are not restricted to the special form of $\tilde{\mathbb{E}}_h$ and apply to general intersection spaces that arise in the context of the L^p - L^q -approach to initial-boundary value problems with inhomogeneous symbols as considered in [15, 28]. We can further allow temporal weights as in [35] in order to obtain flexibility for the initial regularity. The weighted approach is very useful when studying the long-time behavior of solutions (see e.g. [40]). For instance, the arguments from the proof above allow to determine the precise temporal trace space of

$$\begin{aligned} & F_{p,q}^{1+\kappa_0}(\mathbb{R}, w; L^q(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}, w; B_{q,q}^{l+2m\kappa_0}(\mathbb{R}^{d-1})) \\ & \cap H^{1,p}(\mathbb{R}, w; B_{q,q}^{2m\kappa_0}(\mathbb{R}^{d-1})) \cap \bigcap_{j=0}^m F_{p,q}^{\kappa_j}(\mathbb{R}, w; H^{k_j,q}(\mathbb{R}^{d-1})), \end{aligned}$$

where $w(t) = |t|^\gamma$ with $\gamma \in (-1, p-1)$, which is the space of the boundary unknown in the L^p - L^q -approach to boundary value problems of relaxation type (see [15, Section 2]). Here the numbers κ_j and k_j are determined by the orders of the differential operators involved in the problem.

Remark 9.5. Arguing as in Remark 9.2, for $\frac{2p}{p+1} < q < 2p$ the assertions of Proposition 9.3 remain true if one replaces $\tilde{\mathbb{E}}_h$ by \mathbb{E}_h .

Remark 9.6. If temporal traces exist, then compatibility conditions for the data are required to obtain a strong solution $(u, h) \in \mathbb{E}_u \times \mathbb{E}_h$ of (9.1). If $1 - 1/(2q) > 1/p$, then the static boundary conditions imply that $\llbracket u_0 \rrbracket = 0$ and $g_1|_{t=0} = u_0 + \Delta' h_0$ necessarily hold as well. In case $1/2 - 1/(2q) > 1/p$ also $\partial_t h|_{t=0}$ exists, and thus the dynamic boundary condition implies that $g_2|_{t=0} + \llbracket \partial_\nu u_0 \rrbracket$ enjoys at least the same regularity as $\partial_t h|_{t=0}$, i.e., that $g_2|_{t=0} + \llbracket \partial_\nu u_0 \rrbracket \in X_{\partial_t h} = B_{q,p}^{2-2/q-4/p}(\mathbb{R}^{d-1})$. Observe that this condition is not trivial: if g_2 is an arbitrary element of \mathbb{F}_2 , then the trace theorem only gives $g_2|_{t=0} \in B_{q,p}^{1-1/q-2/p}(\mathbb{R}^{d-1})$, which is only half of the required smoothness.

After these considerations we can extend [28, Theorem 7.34] to nontrivial initial values and to show maximal L^p - L^q -regularity for the fully inhomogeneous problem (9.1). For the convenience of the reader we recall the spaces

$$\begin{aligned} \mathbb{E}_0 &= L^p(\mathbb{R}_+; L^q(\dot{\mathbb{R}}^d)), \quad \mathbb{E}_u = H^{1,p}(\mathbb{R}_+; L^q(\dot{\mathbb{R}}^d)) \cap L^p(\mathbb{R}_+; H^{2,q}(\dot{\mathbb{R}}^d)), \quad X_u = B_{q,p}^{2-2/p}(\dot{\mathbb{R}}^d), \\ \mathbb{F}_1 &= F_{p,q}^{1-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{2-1/q}(\mathbb{R}^{d-1})), \\ \mathbb{F}_2 &= F_{p,q}^{1/2-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{1-1/q}(\mathbb{R}^{d-1})), \\ \mathbb{E}_h &= F_{p,q}^{3/2-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap F_{p,q}^{1-1/(2q)}(\mathbb{R}_+; H^{2,q}(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{4-1/q}(\mathbb{R}^{d-1})), \end{aligned}$$

and further the trace spaces X_h and $X_{\partial_t h}$ determined in Proposition 9.3.

Theorem 9.7. *Let $p \in (1, \infty)$ such that $\frac{2p}{p+1} < q < 2p$ and $1 - 1/(2q), 1/2 - 1/(2q) \neq 1/p$, and let $\mu > 0$. Then there is a unique strong solution $(u, h) \in \mathbb{E}_u \times \mathbb{E}_h$ of (9.1) if and only if*

$$f \in \mathbb{E}_0, \quad h_1 \in \mathbb{F}_1, \quad h_2 \in \mathbb{F}_2, \quad u_0 \in X_u, \quad h_0 \in X_h,$$

and the compatibility conditions

$$\llbracket u_0 \rrbracket = 0, \quad g_1|_{t=0} = u_0 - \Delta' h_0 \quad \text{if } 1 - 1/(2q) > 1/p,$$

$$g_2|_{t=0} + \llbracket \partial_\nu u_0 \rrbracket \in X_{\partial_t h} \quad \text{if } 1/2 - 1/(2q) > 1/p,$$

are satisfied. There is a constant $C > 0$, which is independent of the data, such that

$$\|u\|_{\mathbb{E}_u} + \|h\|_{\mathbb{E}_h} \leq C(\|f\|_{\mathbb{E}_0} + \|h_1\|_{\mathbb{F}_1} + \|h_2\|_{\mathbb{F}_2} + \|u_0\|_{X_u} + \|h_0\|_{X_h}).$$

Remark 9.8. (i) In case $p = q$ we precisely recover the result of [18, Theorem 6.1].

(ii) Theorem 9.7 is the basic ingredient to treat the original two-phase Stefan problem with Gibbs-Thomson correction [18, Problem (1.3)] in an L^p - L^q -setting.

- (iii) The restriction $\frac{2p}{p+1} < q < 2p$ is due to the results in [28] and, as indicated there, should not be essential. If one removes this condition in the results of [28], then the combination with our trace results from Proposition 9.3 gives maximal L^p - L^q regularity for all $p, q \in (1, \infty)$, with the same arguments as given in the sequel.

Proof of Theorem 9.7. Step 1. The necessity of the regularity of the data and the compatibility conditions are a consequence of the previous considerations and the Remarks 9.2, 9.5 and 9.6. Further, uniqueness of solutions follows from the homogeneous case [28, Theorem 7.34].

Step 2. We claim that for all $f \in \mathbb{E}_0$ and $u_0 \in X_u$ with $\llbracket u_0 \rrbracket = 0$ if $1 - 1/p > 1/(2q)$ there is $\tilde{u} \in \mathbb{E}_u$ satisfying

$$(9.2) \quad \mu u + \partial_t \tilde{u} - \Delta \tilde{u} = f \quad \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^d, \quad \llbracket \tilde{u} \rrbracket = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{d-1}, \quad \tilde{u}|_{t=0} = u_0 \quad \text{in } \dot{\mathbb{R}}^d.$$

To see this, extend $u_0|_{\mathbb{R}_+^d}$ to $\tilde{u}_0 \in B_{q,p}^{2-2/p}(\mathbb{R}^d)$, and use [14, Proposition 6.1] to define $w_- \in H^{1,p}(\mathbb{R}_+; L^q(\mathbb{R}_+^d)) \cap L^p(\mathbb{R}_+; H^{2,q}(\mathbb{R}_+^d))$ as the restriction to \mathbb{R}_+^d of the unique solution of

$$\mu v + \partial_t v - \Delta v = f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \quad v|_{t=0} = \tilde{u}_0 \quad \text{in } \mathbb{R}^d.$$

Next, use [14, Proposition 6.4] and that $u_0|_{\mathbb{R}_+^d} = w_-|_{\mathbb{R}^{d-1}, t=0} = u_0|_{\mathbb{R}_+^d}$ if $1 - 1/p > 1/(2q)$ to define $w_+ \in H^{1,p}(\mathbb{R}_+; L^q(\mathbb{R}_+^d)) \cap L^p(\mathbb{R}_+; H^{2,q}(\mathbb{R}_+^d))$ as the unique solution of

$$\mu v + \partial_t v - \Delta v = f|_{\mathbb{R}_+^d} \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+^d, \quad v|_{\mathbb{R}^{d-1}} = w_-|_{\mathbb{R}^{d-1}} \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{d-1}, \quad v|_{t=0} = u_0|_{\mathbb{R}_+^d} \quad \text{in } \mathbb{R}_+^d.$$

Now the function $\tilde{u} \in \mathbb{E}_u$, defined by $\tilde{u}|_{\mathbb{R}_+^d} = w_\pm$, satisfies (9.2).

Step 3. By Remark 9.5 there is $\tilde{h} \in \mathbb{E}_h$ such that $\tilde{h}|_{t=0} = h_0$ and $\partial_t \tilde{h}|_{t=0} = g_2|_{t=0} + \llbracket \partial_\nu u_0 \rrbracket$ if $1/2 - 1/(2q) > 1/p$. For \tilde{u} and \tilde{h} we consider the problem

$$(9.3) \quad \begin{cases} \mu w + \partial_t w - \Delta w = 0 & \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^d, \\ \llbracket w \rrbracket = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}^{d-1}, \\ w + \Delta' \sigma = g_1 - \tilde{u} - \Delta' \tilde{h} & \text{on } \mathbb{R}_+ \times \mathbb{R}^{d-1}, \\ \partial_t \sigma - \llbracket \partial_\nu w \rrbracket = g_2 - \partial_t \tilde{h} + \llbracket \partial_\nu \tilde{u} \rrbracket & \text{on } \mathbb{R}_+ \times \mathbb{R}^{d-1}, \\ w|_{t=0} = 0 & \text{in } \dot{\mathbb{R}}^d, \\ \sigma|_{t=0} = 0, & \text{on } \mathbb{R}^{d-1}. \end{cases}$$

Since $\tilde{u}|_{\mathbb{R}^{d-1}} + \Delta' \tilde{h} \in \mathbb{F}_1$ by [14, Proposition 6.4] and Remark 9.5, and further

$$g_1|_{t=0} - \tilde{u}|_{\mathbb{R}^{d-1}, t=0} - \Delta' \tilde{h}|_{t=0} = 0 \quad \text{if } 1/2 - 1/(2q) > 1/p,$$

by assumption, it follows from [28, Proposition 5.14] that

$$g_1 - \tilde{u}|_{\mathbb{R}^{d-1}} - \Delta' \tilde{h} \in {}_0F_{p,q}^{1-1/(2q)}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{2-1/p}(\mathbb{R}^{d-1})),$$

where ${}_0F_{p,q}^\beta$ denotes vanishing traces at $t = 0$. We further have $\partial_t \tilde{h} - \llbracket \partial_\nu \tilde{u} \rrbracket \in \mathbb{F}_2$ by Remark 9.5 and [14, Proposition 6.4], and as before it follows from $g_2|_{t=0} - \partial_t \tilde{h}|_{t=0} + \llbracket \partial_\nu u \rrbracket|_{t=0} = 0$ if $1/2 - 1/(2q) > 1/p$ that

$$g_2 - \partial_t \tilde{h} + \llbracket \partial_\nu \tilde{u} \rrbracket \in {}_0F_{p,q}^{1/2-1/2p}(\mathbb{R}_+; L^q(\mathbb{R}^{d-1})) \cap L^p(\mathbb{R}_+; B_{q,q}^{1-1/p}(\mathbb{R}^{d-1})).$$

Hence [28, Theorem 7.34] provides a solution $(w, \sigma) \in \mathbb{E}_u \times \mathbb{E}_h$ of (9.3). Now $(w + \tilde{u}, \sigma + \tilde{h}) \in \mathbb{E}_u \times \mathbb{E}_h$ is the solution of (9.1). The asserted estimate of (u, h) in terms of the data follows from the estimates in [14, 28] for partially homogeneous problems from above and the continuity of the extension operator from Proposition 9.3. \square

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